

# FRAMES AND OVERSAMPLING FORMULAS FOR BAND LIMITED FUNCTIONS

VINCENZA DEL PRETE

**ABSTRACT.** In this article we obtain families of frames for the space  $B_\omega$  of functions with band in  $[-\omega, \omega]$  by using the theory of shift-invariant spaces. Our results are based on the Gramian analysis of A. Ron and Z. Shen and a variant, due to Bownik, of their characterization of families of functions whose shifts form frames or Riesz bases. We give necessary and sufficient conditions for the translates of a finite number of functions (generators) to be a frame or a Riesz basis for  $B_\omega$ . We also give explicit formulas for the dual generators and we apply them to Hilbert transform sampling and derivative sampling. Finally, we provide numerical experiments which support the theory.

## 1. INTRODUCTION

In many signal and image processing applications, images and signals are assumed to be band limited. A band limited signal is a function which belongs to the space  $B_\omega$  of functions in  $L^2(\mathbb{R})$  whose Fourier transforms have support in  $[-\omega, \omega]$ . Functions belonging to this space can be represented by the Whittaker-Kotelnikov-Shannon series, which is the expansion in terms of the orthonormal basis of translates of the *sinc* function. The coefficients of the expansion are the samples of the function at a uniform grid on  $\mathbb{R}$ , with “density”  $\frac{\omega}{\pi}$ . This sampling density is usually called the *Nyquist* density.

The theory has been extended in many directions by several authors. In one of these extensions the sinc orthonormal basis has been replaced by Riesz bases (see for example [Hi] by J. R. Higgins) or frames. Frames generally are overcomplete and their expansion coefficients are not unique. Their redundancy is useful in applications because the reconstruction is more stable with respect to errors in the calculation of the coefficients and it allows the recovery of missing samples [F]. In signal analysis frames can be viewed as an “oversampling” with respect to the Nyquist density.

The second extension consist in using more than one function to generate the space  $B_\omega$ . In this case the Riesz basis or the frame are formed by the translates of a finite family of functions and the expansion formula is called a multi-channel sampling formula [P] [Hi1].

Finally we mention that there is a huge literature on the problem of reconstruction of signals from non-uniform samples. Since this paper is only concerned with uniform sampling, we refer the reader to the recent article of A. Aldroubi and K. Gröchenig [AG] and the references given there for an extensive review of the problem of non-uniform sampling.

---

*Key words and phrases.* frame, Riesz basis, shift-invariant space, sampling formulas, band limited functions.

In this paper we construct multi-channel uniform sampling formulas for band-limited functions using the theory of frames for shift-invariant spaces. A  $t_o$ -shift-invariant space is a subspace of  $L^2(\mathbb{R})$  that is invariant under all translations  $\tau_{kt_o}, k \in \mathbb{Z}$ , by integer multiples of a positive number  $t_o$ . We recall that  $B_\omega$  is  $t_o$ -shift-invariant for any  $t_o$ . A set  $\Phi$  in a  $t_o$ -shift-invariant space  $S$  is called a set of generators if  $S$  is the closure of the space generated by the family  $E_{\Phi, t_o} = \{\tau_{kt_o} \varphi, \varphi \in \Phi, k \in \mathbb{Z}\}$ . The space  $S$  is said to be finitely generated if it has a finite set of generators. Finitely generated shift-invariant spaces can have different sets of generators; the smallest number of generators is called the length of the space. The structure of finitely generated shift-invariant spaces was investigated by C. de Boor, R. DeVore and A. Ron with the use of fiberization techniques based on the range function [BDR]. These authors gave conditions under which a finitely generated shift-invariant space has a generating set satisfying properties like stability and orthogonality. Successively, A. Ron and Z. Shen introduced the Gramian analysis and extended the results of [BDR] to countable generated SI spaces [RS]. In their paper they characterized sets of generators whose translates form Bessel sequences, frames and Riesz bases. For finitely generated spaces these conditions are expressed in terms of the eigenvalues of the Gramian matrix. In concrete cases it would be useful to have more explicit conditions expressed in terms of the generators or their Fourier transforms. In this paper, using a result of M. Bownik, we obtain these more explicit conditions for the space  $B_\omega$  [B]. We also give explicit formulas for the Fourier transforms of the dual generators.

In the last part of the paper we use these results to obtain multichannel oversampling formulas for band limited signals and we apply them to the derivative sampling. Finally, we support the theory with some numerical experiments.

The paper is organized as follows. In Section 2 we collect some results on frames for shift-invariant spaces and find the range of  $B_\omega$  as a  $t_o$ -shift-invariant space.

In Section 3 we consider a family  $\Phi = \{\varphi_1, \varphi_2, \dots, \varphi_N\}$  of  $N$  generators, where  $N$  is the length of  $B_\omega$  as a  $t_o$ -shift-invariant space. We give a necessary and sufficient condition for  $E_{\Phi, t_o}$  to be a frame or a Riesz basis for  $B_\omega$ . The condition is expressed in terms of the pre-Gramian. To prove this condition we use a result of Bownik which characterizes the system of translates  $E_{\Phi, t_o}$  as being a frame or a Riesz family in terms of “fibers” [B].

In Section 4 we find the family  $\Phi^*$  of dual generators. Here we use the fact that, in the fibered representation of  $B_\omega$ , the frame operator is unitarily equivalent to the operator of multiplication by the dual Gramian matrix  $\tilde{G}_{\Phi, t_o}(x) = J_{\Phi, t_o}(x) J_{\Phi, t_o}^*(x)$  acting on the fiber over  $x$ . The problem of finding the dual generators is reduced to solving the matricial equation  $J_{\Phi, t_o}(x) = J_{\Phi, t_o}(x) J_{\Phi^*, t_o}^*(x) J_{\Phi^*, t_o}(x)$  in the unknown  $J_{\Phi^*, t_o}(x)$  in  $[0, h]$ . Where  $J_{\Phi, t_o}(x)$  is invertible, the solution is the inverse of  $J_{\Phi, t_o}(x)$ , elsewhere it is given by the Moore-Penrose inverse of  $J_{\Phi, t_o}(x)$ . We give the expressions of the Fourier transforms of the dual generators as cross products of translates of the vector  $\hat{\Phi} = (\hat{\varphi}_1, \hat{\varphi}_2, \dots, \hat{\varphi}_N)$ .

In Section 5 we apply the results of the previous sections to obtain two and three-channel sampling formulas for functions of  $B_\omega$  and we apply them to derivative sampling. We also provide some numerical experiments.

## 2. PRELIMINARIES

In this section we collect some results on frames for shift-invariant spaces to be used later. We begin by introducing some notation. The Fourier transform of a function  $f$  in  $L^1(\mathbb{R})$  is

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int f(t)e^{-it\xi} dt.$$

The convolution of two functions  $f$  and  $g$  is

$$f * g(x) = \frac{1}{\sqrt{2\pi}} \int f(x-y)g(y) dy,$$

so that  $\mathcal{F}(f * g) = \mathcal{F}f \mathcal{F}g$ . Let  $h$  be a positive real number;  $L_h^p$  is the space of  $h$ -periodic functions on  $\mathbb{R}$  such that

$$(2.1) \quad \|f\|_{L_h^p} = \left( \frac{1}{h} \int_0^h |f(x)|^p dx \right)^{1/p} < \infty.$$

With the symbol  $\ell^2(\mathbb{Z}; \mathbb{C}^N)$  we shall denote the space of  $\mathbb{C}^N$ -valued sequences  $c = (c(n))_{n \in \mathbb{Z}}$  such that

$$\|c\|_{\ell_2} = \left( \sum_{n \in \mathbb{Z}} |c(n)|^2 \right)^{1/2} < \infty.$$

Let  $H$  be a subspace of  $L^2(\mathbb{R})$ . Given a subset  $\Phi = \{\varphi_j, j = 1, \dots, N\}$  of  $H$  and a positive number  $t_o$  denote by  $E_{\Phi, t_o}$  the set

$$E_{\Phi, t_o} = \{\tau_{nt_o} \varphi_j, n \in \mathbb{Z}, j = 1, \dots, N\}.$$

Here  $\tau_a f(x) = f(x+a)$ . The closure of the space generated by  $E_{\Phi, t_o}$  will be denoted by  $S_{\Phi, t_o}$ . The family  $E_{\Phi, t_o}$  is a frame for  $H$  if the operator  $T_{\Phi, t_o} : \ell^2(\mathbb{Z}; \mathbb{C}^N) \rightarrow H$  defined by

$$T_{\Phi, t_o} c = \sum_{j=1}^N \sum_{n \in \mathbb{Z}} c_j(n) \tau_{nt_o} \varphi_j$$

is continuous, surjective and  $\text{ran}(T_{\Phi, t_o})$  is closed. The family  $E_{\Phi, t_o}$  is a frame for  $H$  if and only if there exist two constants  $0 < A \leq B$  such that

$$(2.2) \quad A\|f\|^2 \leq \sum_{j=1}^N \sum_{n \in \mathbb{Z}} |\langle f, \tau_{nt_o} \varphi_j \rangle|^2 \leq B\|f\|^2 \quad \forall f \in H.$$

The constants  $A$  and  $B$  are called frame bounds. If  $A = B$  the frame is called *tight* and if  $A = B = 1$  a *Parseval* frame. Denote by  $T_{\Phi, t_o}^* : H \rightarrow \ell^2(\mathbb{Z}; \mathbb{C}^N)$  the adjoint of  $T_{\Phi, t_o}$ , defined by

$$(2.3) \quad (T_{\Phi, t_o}^* f)_j(n) = \langle f, \tau_{nt_o} \varphi_j \rangle \quad n \in \mathbb{Z}, j = 1, \dots, N.$$

The operator  $T_{\Phi, t_o}^* T_{\Phi, t_o} : H \rightarrow H$  is called *frame operator*. The set  $E_{\Phi, t_o}$  is a frame for  $H$  if and only if the frame operator is continuously invertible and

$$T_{\Phi, t_o} T_{\Phi, t_o}^* f = \sum_{j=1}^N \sum_{n \in \mathbb{Z}} \langle f, \tau_{nt_o} \varphi_j \rangle \tau_{nt_o} \varphi_j \quad f \in H.$$

Observe that (2.2) can be written  $AI \leq T_{\Phi, t_o} T_{\Phi, t_o}^* \leq BI$ , where  $I$  is the identity operator on  $H$ . Denote by  $\Phi^*$  the family  $\Phi^* = \{\varphi_j^*, j = 1, \dots, N\}$ , where

$$(2.4) \quad \varphi_j^* = (T_{\Phi, t_o} T_{\Phi, t_o}^*)^{-1} \varphi_j \quad 1 \leq j \leq N.$$

If  $E_{\Phi, t_o}$  is a frame for  $H$  then  $E_{\Phi^*, t_o}$  is also a frame (the *dual frame*), and  $T_{\Phi, t_o} T_{\Phi^*, t_o}^* = T_{\Phi^*, t_o}^* T_{\Phi, t_o} = I$ . Explicitly

$$(2.5) \quad f = \sum_{j=1}^N \sum_{n \in \mathbb{Z}} \langle f, \tau_{nt_o} \varphi_j^* \rangle \tau_{nt_o} \varphi_j = \sum_{j=1}^N \sum_{n \in \mathbb{Z}} \langle f, \tau_{nt_o} \varphi_j \rangle \tau_{nt_o} \varphi_j^* \quad \forall f \in H.$$

The elements of  $\Phi$  are called *generators* and the elements of  $\Phi^*$  *dual generators*. If the family  $E_{\Phi, t_o}$  is a frame for  $H$  and the operator  $T_{\Phi, t_o}$  is injective, then  $E_{\Phi, t_o}$  is called a *Riesz basis*.

In what follows  $t_o$  is a positive parameter. To simplify notation, throughout the paper we shall set

$$h = \frac{2\pi}{t_o}.$$

A subspace  $S$  of  $L^2(\mathbb{R})$  is  $t_o$ -shift-invariant if it is invariant under all translations by a multiple of  $t_o$ . The following bracket product plays an important role in Ron and Shen's analysis of shift-invariant spaces. For  $f$  and  $g \in L^2(\mathbb{R})$ , define

$$(2.6) \quad [f, g] = h \sum_{j \in \mathbb{Z}} f(\cdot + jh) \bar{g}(\cdot + jh).$$

Note that  $[f, g]$  is in  $L_h^1$  and  $\|[f, f]\|_{L_h^1} = \|f\|_2^2$ . The Fourier coefficients of  $[\hat{f}, \hat{g}]$  are given by

$$(2.7) \quad [\hat{f}, \hat{g}]^\wedge(\ell) = \langle f, \tau_{\ell t_o} g \rangle \quad \ell \in \mathbb{Z}.$$

Indeed

$$[\hat{f}, \hat{g}]^\wedge(\ell) = \int_0^h \sum_j \tau_{jh}(\hat{f} \bar{\hat{g}})(x) e^{-2\pi i \ell \frac{x}{h}} dx = \int \hat{f}(x) \bar{\hat{g}}(x) e^{-i \ell t_o x} dx$$

If  $S$  is a  $t_o$ -shift-invariant space and there exists a finite family  $\Phi$  such that  $S = S_{\Phi, t_o}$ , then we say that  $S$  is finitely generated. Riesz bases for finitely generated shift-invariant spaces have been studied by various authors. In [BDR] the authors give a characterization of such bases. A characterization of frames and tight frames also for countable sets  $\Phi$  has been given by Ron and Shen in [RS1]. The principal notions of their theory are the *pre-Gramian*, the *Gramian* and the *dual Gramian* matrices.

The pre-Gramian  $J_{\Phi, t_o}$  is the  $h$ -periodic function mapping  $\mathbb{R}$  to the space of  $\infty \times N$ -matrices defined on  $[0, h]$  by

$$(2.8) \quad (J_{\Phi, t_o})_{j\ell}(x) = \sqrt{h} \widehat{\varphi}_\ell(x + jh), \quad j \in \mathbb{Z}, \ell = 1, \dots, N.$$

The pre-Gramian  $J_{\Phi, t_o}$  should not be confused with the matrix-valued function whose entries are  $\sqrt{h} \widehat{\varphi}_\ell(x + jh)$ , for all  $x \in \mathbb{R}$ , which is not periodic. The spectrum of the space  $S_{\Phi, t_o}$  is defined as

$$(2.9) \quad \sigma(S_{\Phi, t_o}) = \{x \in \mathbb{R} : J_{\Phi, t_o}(x) \neq 0\}$$

or, equivalently, as the support of  $\sum_{j=1}^N [\widehat{\varphi}_j, \widehat{\varphi}_j]$ . Of course, since the functions  $\widehat{\varphi}_j$  are defined only up to a null-set, the support is intended in the sense of distributions, i.e. as the complement of the largest open set on which the function  $\sum_{j=1}^N [\widehat{\varphi}_j, \widehat{\varphi}_j]$  vanishes as distribution. It was proved in [BDR] that the spectrum of a finitely generated space depends only on the space itself and not on the particular selection of its generators.

Denote by  $J_{\Phi, t_o}^*$  the adjoint of  $J_{\Phi, t_o}$ . The *Gramian* matrix  $G_{\Phi, t_o} = J_{\Phi, t_o}^* J_{\Phi, t_o}$  is the  $N \times N$  matrix whose elements are the  $h$ -periodic functions

$$(2.10) \quad (G_{\Phi, t_o})_{j\ell} = [\widehat{\varphi_\ell}, \widehat{\varphi_j}].$$

The *dual Gramian*  $\tilde{G}_{\Phi, t_o} = J_{\Phi, t_o} J_{\Phi, t_o}^*$  is the infinite matrix whose elements are

$$(2.11) \quad (\tilde{G}_{\Phi, t_o})_{j\ell} = h \sum_{n=1}^N \tau_{jh} \widehat{\varphi_n} \tau_{\ell h} \overline{\widehat{\varphi_n}}, \quad j, \ell \in \mathbb{Z}.$$

The importance of these two matrices lies in the fact that the Gramian matrix represents the operator  $T_{\Phi, t_o}^* T_{\Phi, t_o}$  and the dual Gramian represents the operator  $T_{\Phi, t_o} T_{\Phi, t_o}^*$  and many properties of these operators can be studied by looking at them. Indeed, by the theory of Ron and Shen [RS], after conjugating with an isometry, the operator  $T_{\Phi, t_o} T_{\Phi, t_o}^*$  can be decomposed into a measurable field of operators, acting on  $\ell^2(\mathbb{Z})$ , which are represented by the dual Gramian matrix  $\tilde{G}_{\Phi, t_o}$  in the canonical basis (see formulas (3.4) and (3.5) below). Similarly, after conjugation with a Fourier transform, the operator  $T_{\Phi, t_o}^* T_{\Phi, t_o}$  is represented by the Gramian  $G_{\Phi, t_o}$ .

Denote by  $L_h^2(\mathbb{R}; \ell^2(\mathbb{Z}))$  the Hilbert space of  $\ell^2(\mathbb{Z})$ -valued  $h$ -periodic functions on  $\mathbb{R}$  such that

$$(2.12) \quad \|f\|_{L_h^2(\mathbb{R}; \ell^2(\mathbb{Z}))} = \left( \frac{1}{h} \int_0^h \|f(x)\|_{\ell^2}^2 dx \right)^{\frac{1}{2}} < \infty.$$

For every  $f$  in  $L^2(\mathbb{R})$  we denote by  $\mathcal{L}_h f$  the  $\ell^2(\mathbb{Z})$ -valued function defined on  $[0, h]$  by

$$(2.13) \quad \mathcal{L}_h f(x) = \sqrt{h} \sum_{\ell \in \mathbb{Z}} f(x + \ell h) \delta_\ell \quad x \in [0, h]$$

and extended to  $\mathbb{R}$  as a periodic function of period  $h$ . Here  $\{\delta_\ell : \ell \in \mathbb{Z}\}$  is the canonical basis of  $\ell^2(\mathbb{Z})$ . The map  $f \mapsto \mathcal{L}_h f$  is an isometry of  $L^2(\mathbb{R})$  onto  $L_h^2(\mathbb{R}; \ell^2(\mathbb{Z}))$ . Observe that the vectors  $\mathcal{L}_h \widehat{\varphi_j}$ ,  $j = 1, \dots, N$  are the columns of the pre-Gramian  $J_{\Phi, t_o}$ , i.e.

$$(2.14) \quad J_{\Phi, t_o} = (\mathcal{L}_h \widehat{\varphi_1}, \dots, \mathcal{L}_h \widehat{\varphi_N}).$$

The map  $\mathcal{L}_h$  links  $t_o$ -shift-invariant subspaces of  $L^2(\mathbb{R})$  with  $h$ -doubly-invariant subspaces of  $L_h^2(\mathbb{R}; \ell^2(\mathbb{Z}))$ . We recall that a subspace of  $L_h^2(\mathbb{R}; \ell^2(\mathbb{Z}))$  is  $h$ -doubly-invariant if it is invariant under pointwise multiplication by  $e^{2\pi i k \frac{x}{h}}$ ,  $k \in \mathbb{Z}$ . Obviously a subspace  $S$  of  $L^2(\mathbb{R})$  is  $t_o$ -shift-invariant if and only if the space

$$(2.15) \quad \mathcal{L}_h(\hat{S}) = \{\mathcal{L}_h \hat{f}, f \in S\}$$

is  $h$ -doubly-invariant. T.P. Srinivasan gave a characterization of doubly-invariant spaces in terms of *range functions* (see [H],[S]). As remarked by de Boor, DeVore and Ron [BDR] a similar characterization of shift-invariant spaces follows from it (see Proposition 2.1 below). In our context a range function is a  $h$ -periodic map  $\mathcal{R}$  from  $\mathbb{R}$  to the closed subspaces of  $\ell^2(\mathbb{Z})$ . The function  $\mathcal{R}$  is measurable if the map  $\mathcal{P}$  which maps a point  $x \in \mathbb{R}$  to the orthogonal projection  $\mathcal{P}(x)$  onto  $\mathcal{R}(x)$  is weakly measurable in the operator sense, i.e. the function

$$x \mapsto (P(x)\varphi, \psi)_{\ell^2(\mathbb{Z})}$$

is measurable for all  $\varphi$  and  $\psi \in \ell^2(\mathbb{Z})$ . Range functions which are equal almost everywhere are identified.

**Proposition 2.1.** *A closed subspace  $S$  of  $L^2(\mathbb{R})$  is  $t_o$ -shift-invariant if and only if*

$$(2.16) \quad S = \{f \in L^2(\mathbb{R}) : \mathcal{L}_h \hat{f}(x) \in \mathcal{R}(x) \text{ for a.e. } x \in \mathbb{R}\}$$

for some measurable  $h$ -periodic range function  $\mathcal{R}$ .

Obviously  $t_o$ -shift-invariant subspaces with the same range function coincide. This observation shall be used in the proof of Theorems 3.6 and 3.7. In Theorem 2.2 below we compute the range function of the space of band-limited functions

$$B_\omega = \{f \in L^2(\mathbb{R}) : \text{supp}(\hat{f}) \subset [-\omega, \omega]\}.$$

In general it is not a simple matter to compute the range function of a space. Fortunately, if the space  $S$  is finitely generated, the range function can be written in terms of the generators; indeed in [BDR] it was shown that if  $S = S_{\Phi, t_o}$  and  $\Phi = \{\varphi_j, \quad j = 1, \dots, N\}$  then

$$(2.17) \quad \mathcal{R}_h(S_{\Phi, t_o})(x) = \text{span}\{\mathcal{L}_h \widehat{\varphi_j}(x) : j = 1, \dots, N\}$$

is the space generated by the columns of the pre-Gramian matrix  $J_{\Phi, t_o}$ . This result holds also for countable sets of generators. Of course a shift-invariant space  $S$  can have more than one family of generators; the smallest number of generators is called the length of the space

$$(2.18) \quad \text{len}_{t_o}(S) = \min\{\#\Phi, S = S_{\Phi, t_o}\}.$$

In [BDR] it has been proved that if  $S$  is finitely generated then

$$(2.19) \quad \text{len}_{t_o}(S) = \text{ess sup} \{\dim \mathcal{R}_h(S_{\Phi, t_o})(x) : x \in [0, h]\}.$$

To state the next theorem we need some notation. We denote by  $[a]$  the greatest integer less than  $a$ . Recall that  $h = 2\pi/t_o$  and set  $\ell = [\frac{\omega}{h}] + 1$ .

If  $\frac{\omega}{\ell} \leq h < \frac{\omega}{\ell-1}$  then  $0 \leq -\omega + \ell h < \omega - (\ell-1)h < h$ . We denote by  $I_-, I, I_+$  the intervals defined by

$$(2.20) \quad I_- = (0, -\omega + \ell h), \quad I = (-\omega + \ell h, \omega - (\ell-1)h), \quad I_+ = (\omega - (\ell-1)h, h)$$

Similarly if  $\frac{\omega}{\ell-1} \leq h < \frac{\omega}{\ell}$  then  $0 < \omega - (\ell-1)h \leq -\omega + \ell h < h$ ; in this case we denote by  $K_-, K, K_+$  the intervals defined by

$$(2.21) \quad K_- = (0, \omega - (\ell-1)h), \quad K = (\omega - (\ell-1)h, -\omega + \ell h), \quad K_+ = (-\omega + \ell h, h).$$

Recall that  $\{\delta_\ell : \ell \in \mathbb{Z}\}$  is the canonical basis of  $\ell^2(\mathbb{Z})$ .

**Theorem 2.2.** *Let  $t_o$  be a positive parameter,  $h = 2\pi/t_o$  and set  $\ell = [\frac{\omega}{h}] + 1$ . Then*

(i) *if  $\frac{\omega}{\ell} \leq h < \frac{\omega}{\ell-1}$  the range function of the space  $B_\omega$  is*

$$(2.22) \quad \mathcal{R}_h(B_\omega)(x) = \begin{cases} \text{span}\{\delta_j : -(\ell-1) \leq j \leq \ell-1\}, & x \in I_- \\ \text{span}\{\delta_j : -\ell \leq j \leq \ell-1\}, & x \in I \\ \text{span}\{\delta_j : -\ell \leq j \leq \ell-2\}, & x \in I_+. \end{cases}$$

*Note that if  $h = \frac{\omega}{\ell}$  then the intervals  $I_-$  and  $I_+$  are empty.*

(ii) If  $\frac{\phi}{\ell-1} \leq h < \frac{\phi}{\ell-2}$  the range function of the space  $B_\omega$  is

$$(2.23) \quad \mathcal{R}_h(B_\omega)(x) = \begin{cases} \text{span}\{\delta_j : -(\ell-1) \leq j \leq \ell-1\}, & x \in K_- \\ \text{span}\{\delta_j : -(\ell-1) \leq j \leq \ell-2\}, & x \in K \\ \text{span}\{\delta_j : -\ell \leq j \leq \ell-2\}, & x \in K_+. \end{cases}$$

Note that if  $h = \frac{\phi}{\ell-1}$  then the interval  $K$  is empty.

*Proof.* For the sake of simplicity we prove the theorem only for  $\ell = 2$ , i.e.  $\phi/2 \leq h < \phi$ . The proof in the other cases is similar. Denote by  $\mathcal{M}$  the space of all functions  $g$  in  $L_h^2(\mathbb{R}; \ell^2(\mathbb{Z}))$  such that  $g(x) \in \mathcal{R}(B_\omega)(x)$  for a.e.  $x \in [0, h]$ . Let  $\mathbf{Q}$  denote the orthogonal projection of  $L_h^2(\mathbb{R}; \ell^2(\mathbb{Z}))$  onto  $\mathcal{M}$ . By [H, Lemma p. 58] for  $g \in L_h^2(\mathbb{R}; \ell^2(\mathbb{Z}))$  we have

$$(2.24) \quad \mathbf{Q}g(x) = Q(x)g(x) \quad \text{a.e. } x$$

where  $Q(x)$  is the orthogonal projection of  $\ell^2(\mathbb{Z})$  onto  $\mathcal{R}(B_\omega)(x)$ . Thus, to determine  $\mathcal{R}(B_\omega)(x)$  we only need to describe the projection  $Q(x)$  for a.e.  $x$ . Denote by  $\Lambda_\phi$  the space of functions in  $L^2(\mathbb{R})$  with support in  $[-\phi, \phi]$  and by  $\mathbf{P} : L^2(\mathbb{R}) \mapsto \Lambda_\phi$  the orthogonal projection onto  $\Lambda_\phi$ , that is  $\mathbf{P}f = \chi_{[-\phi, \phi]}f$ . By Proposition 2.1  $\mathcal{L}_h(\Lambda_\phi) = \mathcal{L}_h(\widehat{B_\omega}) = \mathcal{M}$ . Thus we have

$$(2.25) \quad \mathbf{Q} = \mathcal{L}_h \mathbf{P} \mathcal{L}_h^{-1}$$

because  $\mathcal{L}_h$  is an isometry. Let  $\psi = \sum_{n \in \mathbb{Z}} \psi_n \delta_n \in L_h^2(\mathbb{R}; \ell^2(\mathbb{Z}))$ , then  $\mathcal{L}_h^{-1}\psi(x) = \psi_n(x)$  for  $x \in [nh, (n+1)h]$ ,  $n \in \mathbb{Z}$ ; i.e.

$$\mathcal{L}_h^{-1}\psi = \sum_n \chi_{[nh, (n+1)h]} \psi_n.$$

Hence

$$(2.26) \quad \mathbf{P} \mathcal{L}_h^{-1}\psi = \sum_n \chi_{[-\phi, \phi]} \chi_{[nh, (n+1)h]} \psi_n.$$

Since  $\frac{\phi}{2} \leq h < \phi$

$$\chi_{[-\phi, \phi]} \chi_{[nh, (n+1)h]} = \begin{cases} \chi_{[0, h]} & \text{if } n = 0 \\ \chi_{[-h, 0]} & \text{if } n = -1 \\ \chi_{[h, \phi]} & \text{if } n = 1 \\ \chi_{[-\phi, -h]} & \text{if } n = -2 \\ 0 & \text{otherwise.} \end{cases}$$

It follows that

$$(2.27) \quad \mathbf{P} \mathcal{L}_h^{-1}\psi = \chi_{[-\phi, -h]} \psi_{-2} + \chi_{[-h, 0]} \psi_{-1} + \chi_{[0, h]} \psi_0 + \chi_{[h, \phi]} \psi_1.$$

Now we find  $\mathcal{L}_h \mathbf{P} \mathcal{L}_h^{-1}\psi(x)$ .

First suppose that  $h \geq \frac{2}{3}\phi$ ; then  $0 < \phi - h \leq 2h - \phi < h$ . Hence from (2.27) it follows that for each  $\psi \in \ell^2(\mathbb{Z})$

$$\mathcal{L}_h \mathbf{P} \mathcal{L}_h^{-1}\psi(x) = \begin{cases} \psi_{-1}(x) \delta_{-1} + \psi_0(x) \delta_0 + \psi_1(x) \delta_1 & x \in (0, \phi - h) \\ \psi_{-1}(x) \delta_{-1} + \psi_0(x) \delta_0 & x \in (\phi - h, 2h - \phi) \\ \psi_{-2}(x) \delta_{-2} + \psi_{-1}(x) \delta_{-1} + \psi_0(x) \delta_0 & x \in (2h - \phi, h). \end{cases}$$

Suppose now  $h < \frac{2}{3}\varnothing$ ; in this case  $2h - \varnothing < \varnothing - h < h$ . Hence from (2.27) we have  $\psi \in \ell^2(\mathbb{Z})$

$$\mathcal{L}_h \mathbf{P} \mathcal{L}_h^{-1} \psi(x) = \begin{cases} \psi_{-1}(x) \delta_{-1} + \psi_0(x) \delta_0 + \psi_1(x) \delta_1 & x \in (0, 2h - \varnothing) \\ \sum_{j=-2}^1 \psi_j(x) \delta_j & x \in (2h - \varnothing, \varnothing - h) \\ \psi_{-2}(x) \delta_{-2} + \psi_{-1}(x) \delta_{-1} + \psi_0(x) \delta_0 & x \in (\varnothing - h, h). \end{cases}$$

Note that if  $h = \varnothing/2$  the intervals  $(0, 2h - \varnothing)$  and  $(\varnothing - h, h)$  are empty. This determines completely the projection  $Q(x)$  by (2.24) and (2.25).  $\square$

The following corollary is a straightforward consequence of Theorem 2.2.

**Corollary 2.3.** *Let  $\ell = \lceil \frac{\varnothing}{h} \rceil + 1$ . Then the length of  $B_\omega$ , as  $t_o$ -shift-invariant space, is*

$$(2.28) \quad \text{len}_{t_o}(B_\omega) = \begin{cases} 2\ell & \text{if } \frac{\varnothing}{\ell} \leq h < \frac{\varnothing}{\ell-1}, \\ 2\ell - 1 & \text{if } \frac{\varnothing}{\ell-1} \leq h < \frac{\varnothing}{\ell-2}. \end{cases}$$

*Proof.* The thesis follows immediately from (2.19) and Theorem 2.2.  $\square$

### 3. FRAMES FOR THE SPACE $B_\omega$

Let  $\Phi$  be a finite family of generators for  $B_\omega$ . In this section we find conditions under which  $E_{\Phi, t_o}$  is a frame for  $B_\omega$ . First we prove a representation formula for the frame operator analogous to the formula proved by Heil and Walnut for Gabor frames [HW, Theorem 4.2.1]. From this formula we deduce a simple necessary condition (see Proposition 3.2). The formula will also be useful in Section 4 to find the dual generators.

Let  $\Phi = \{\varphi_j, 1 \leq j \leq N\}$  be a family of functions in  $B_\omega$ ,  $t_o$  a positive parameter and  $h = 2\pi/t_o$ . Denote by  $\Omega_{\Phi, t_o}^k$  the function

$$(3.1) \quad \Omega_{\Phi, t_o}^k = h \sum_{j=1}^N \widehat{\varphi_j} \tau_{kh} \overline{\widehat{\varphi_j}} \quad k \in \mathbb{Z}.$$

**Theorem 3.1.** *If  $\sum_k \|\Omega_{\Phi, t_o}^k\|_\infty < \infty$  then the operator  $T_{\Phi, t_o}$  from  $\ell^2(\mathbb{Z}; \mathbb{C}^N)$  to  $B_\omega$  is bounded and*

$$(3.2) \quad \mathcal{F} T_{\Phi, t_o} T_{\Phi, t_o}^* \mathcal{F}^{-1} = \sum_k \Omega_{\Phi, t_o}^k \tau_{kh}$$

on  $L^2(\mathbb{R})$ , where the series converges in the operator norm.

*Proof.* First we show that if  $f \in L^2(\mathbb{R})$  then  $[\widehat{f}, \widehat{\varphi_j}]$  is in  $L_h^2$  for  $1 \leq j \leq N$ . Indeed

$$\begin{aligned} \int_0^h |[\widehat{f}, \widehat{\varphi_j}]|^2 dx &= h \int_0^h \left( \sum_\ell \widehat{f}(x + \ell h) \widehat{\varphi_j}(x + \ell h) \right) [\widehat{f}, \widehat{\varphi_j}](x) dx \\ &= h \sum_\ell \int_{\ell h}^{(\ell+1)h} \widehat{f}(z) \widehat{\varphi_j}(z) [\widehat{f}, \widehat{\varphi_j}](z) dz \\ &= h^2 \int \sum_k \widehat{\varphi_j}(z) \widehat{\varphi_j}(z + kh) \widehat{f}(z + kh) \overline{\widehat{f}}(z) dz. \end{aligned}$$



Note that we may exchange the sum and the integral because  $\widehat{f}\widehat{\varphi_j}$  has compact support and the sum is finite. By summing over  $j$  and exchanging the sums we obtain

$$\sum_{j=1}^N \frac{1}{h} \int_0^h |[\widehat{f}, \widehat{\varphi_j}]|^2 dx = h \int \sum_k \sum_{j=1}^N \widehat{\varphi_j}(z) \overline{\widehat{\varphi_j}(z+kh)} \widehat{f}(z+kh) \overline{\widehat{f}(z)} dz.$$

Therefore, by (3.1)

$$(3.3) \quad \sum_{j=1}^N \frac{1}{h} \int_0^h |[\widehat{f}, \widehat{\varphi_j}]|^2 dx = \langle \sum_k \Omega_{\Phi, t_o}^k \tau_{kh} \widehat{f}, \widehat{f} \rangle.$$

By Schwarz's inequality the right hand side is less than  $\|\widehat{f}\|_2^2 \sum_k \|\Omega_{\Phi, t_o}^k\|_\infty$ . Hence  $[\widehat{f}, \widehat{\varphi_j}] \in L_h^2$ , for  $1 \leq j \leq N$ . By (2.7) the Fourier coefficients of  $[\widehat{f}, \widehat{\varphi_j}]$  are  $\langle f, \tau_{kh} \varphi_j \rangle$ ,  $k \in \mathbb{Z}$ . Hence, by (3.3), Plancherel's formula and (2.3)

$$\begin{aligned} \langle \sum_k \Omega_{\Phi, t_o}^k \tau_{kh} \widehat{f}, \widehat{f} \rangle &= \sum_{j=1}^N \sum_k |\langle f, \tau_{kh} \varphi_j \rangle|^2 \\ &= \|T_{\Phi, t_o}^* f\|^2 \\ &= \langle T_{\Phi, t_o} T_{\Phi, t_o}^* f, f \rangle. \end{aligned}$$

This proves (3.2). Hence  $T_{\Phi, t_o}^*$  is bounded from  $B_\omega$  to  $\ell^2(\mathbb{Z}; \mathbb{C}^N)$  and  $T_{\Phi, t_o}$  is bounded from  $\ell^2(\mathbb{Z}; \mathbb{C}^N)$  to  $B_\omega$ .  $\square$

Now by using formula (3.2) we show that the operator  $T_{\Phi, t_o} T_{\Phi, t_o}^*$  is unitarily equivalent to the operator of multiplication by the matrix  $\widetilde{G}_{\Phi, t_o}$  acting on the space  $L_h^2(\mathbb{R}; \ell^2(\mathbb{Z}))$ . Our proof, given for the sake of completeness, is a simple alternative derivation of a result of Ron and Shen for general shift-invariant spaces [RS]. Let  $f$  be a function in  $L^2(\mathbb{R})$ . By (3.2) for each  $j \in \mathbb{Z}$

$$\begin{aligned} \mathcal{F} T_{\Phi, t_o} T_{\Phi, t_o}^* \mathcal{F}^{-1} f(x+jh) &= \sum_k \Omega_{\Phi, t_o}^k(x+jh) f(x+jh+kh) \\ &= \sum_\ell \Omega_{\Phi, t_o}^{\ell-j}(x+jh) f(x+\ell h). \end{aligned}$$

Now we observe that

$$\Omega_{\Phi, t_o}^{\ell-j}(x+jh) = \overline{\Omega_{\Phi, t_o}^{j-\ell}(x+\ell h)} \quad \text{for a.e. } x \in \mathbb{R} \quad j, \ell \in \mathbb{Z}.$$

Therefore by (2.11) and (3.1) we get

$$\mathcal{F} T_{\Phi, t_o} T_{\Phi, t_o}^* \mathcal{F}^{-1} f(x+jh) = \sum_\ell (\widetilde{G}_{\Phi, t_o})_{j\ell}(x) f(x+\ell h) \quad j, \ell \in \mathbb{Z}.$$

By (2.13) this formula implies that

$$(3.4) \quad \mathcal{L}_h \mathcal{F} T_{\Phi, t_o} T_{\Phi, t_o}^* \mathcal{F}^{-1} f = \widetilde{G}_{\Phi, t_o} \mathcal{L}_h f \quad [0, h].$$

This shows that the operator  $T_{\Phi, t_o} T_{\Phi, t_o}^*$  is unitarily equivalent to the operator  $\widetilde{\mathcal{G}}_{\Phi, t_o}$  defined by

$$(3.5) \quad \widetilde{\mathcal{G}}_{\Phi, t_o} g(x) = \widetilde{G}_{\Phi, t_o}(x) g(x)$$

for almost every  $x \in [0, h]$ . An operator of this form is said to be decomposable into the measurable field  $x \mapsto \tilde{\mathcal{G}}_{\Phi, t_o}(x)$  of operators on  $l^2(\mathbb{Z})$ . We shall use this representation in Section 3 to find the dual generators of frames.

In the following proposition we give a necessary condition for  $E_{\Phi, t_o}$  to be a frame. The proof mimics closely an argument of Heil and Walnut for Gabor frames [HW].

**Proposition 3.2.** *Let  $\Phi = \{\varphi_j, 1 \leq j \leq N\}$  be a family of functions of  $B_\omega$ . If  $E_{\Phi, t_o}$  is a frame for  $B_\omega$  then there exist  $0 < \delta \leq \gamma < \infty$  such that*

$$(3.6) \quad \delta \leq \sum_{j=1}^N |\widehat{\varphi_j}|^2 \leq \gamma \quad \text{a.e. in } [-\phi, \phi].$$

*Proof.* Observe that  $h \sum_{j=1}^N |\widehat{\varphi_j}|^2 = \Omega_{\Phi, t_o}^0$ . We only prove that  $0 < \text{ess inf } \Omega_{\Phi, t_o}^0$  because the proof of the other inequality is analogous. Suppose that  $\text{ess inf } \Omega_{\Phi, t_o}^0 = 0$ ; then for each  $\epsilon$  there exists a set  $E_\epsilon \subset (\phi, \phi)$ , of positive measure, such that  $\Omega_{\Phi, t_o}^0(x) < \epsilon$  for a.e.  $x \in E_\epsilon$ . We may suppose that there exists an interval  $I$  of measure  $h$  such that  $E_\epsilon \subset I$ . Let  $f \in B_\omega$  be defined by  $\hat{f} = \chi_{E_\epsilon}$ ; by the Parseval and the Plancherel formula

$$\sum_k |\langle f, \tau_{kt_o} \varphi_j \rangle|^2 = \sum_k \left| \int_I \hat{f} \overline{\widehat{\varphi_j}} e^{2\pi i k \frac{x}{h}} dx \right|^2 = h \int_I \chi_{E_\epsilon} |\widehat{\varphi_j}|^2 dx \quad j = 1, \dots, N.$$

Therefore

$$\sum_{j=1}^N \sum_k |\langle f, \tau_{kt_o} \varphi_j \rangle|^2 = \int_I \chi_{E_\epsilon}(x) \Omega_{\Phi, t_o}^0(x) dx < \epsilon h \|\chi_{E_\epsilon}\|^2 = \epsilon h \|f\|^2.$$

This contradicts the fact that  $E_{\Phi, t_o}$  is a frame.  $\square$

Next, we give a necessary and sufficient conditions for  $E_{\Phi, t_o}$  to be a frame or a Riesz basis for  $B_\omega$ , when  $\Phi$  is a subset of  $B_\omega$  of cardinality  $\text{len}_{t_o}(B_\omega)$  (see Theorems 3.6 and 3.7 below). Our characterization will be given in terms of the pre-Gramian. Strictly speaking, the pre-Gramian  $J_{\Phi, t_o}$  is an infinite matrix. However we shall see that all but a finite number of the rows of  $J_{\Phi, t_o}$  vanish. Hence we may identify it with a finite matrix. We shall need the following

**Lemma 3.3.** *Let  $h$  be a positive number,  $\ell = \lceil \frac{\phi}{h} \rceil + 1$  and  $g \in B_\omega$ . Let  $I_-, I, I_+$  and  $K_-, K, K_+$  be the intervals defined in the previous section (see (2.20) and (2.21)).*

(i) *Suppose that  $\frac{\phi}{\ell} \leq h < \frac{\phi}{\ell-1}$ . Then  $\tau_{jh} g(x) = 0$*

$$(3.7) \quad \begin{aligned} & \text{if } x \in I_- \quad \text{and } j \notin \{-(\ell-1) \leq j \leq \ell-1\} \\ & \text{if } x \in I \quad \text{and } j \notin \{-\ell \leq j \leq \ell-1\} \\ & \text{if } x \in I_+ \quad \text{and } j \notin \{-\ell \leq j \leq \ell-2\}. \end{aligned}$$

*Note that if  $h = \frac{\phi}{\ell}$  then  $I_-$  and  $I_+$  are empty.*

(ii) *Suppose that  $\frac{\phi}{\ell-1} \leq h < \frac{\phi}{\ell-2}$ . Then  $\tau_{jh} g(x) = 0$*

$$(3.8) \quad \begin{aligned} & \text{if } x \in K_- \quad \text{and } j \notin \{-(\ell-1) \leq j \leq \ell-1\} \\ & \text{if } x \in K \quad \text{and } j \notin \{-(\ell-1) \leq j \leq \ell-2\} \\ & \text{if } x \in K_+ \quad \text{and } j \notin \{-\ell \leq j \leq \ell-2\}. \end{aligned}$$

*Note that if  $h = \frac{\phi}{\ell-1}$  then  $K$  is empty.*

We omit the proof which is straightforward.

We consider separately the two cases  $h < \frac{\phi}{\ell-1}$  and  $\frac{\phi}{\ell-1} \leq h$ . Assume first that  $\frac{\phi}{\ell} \leq h < \frac{\phi}{\ell-1}$ . Then  $\text{len}_{t_o}(B_\omega) = 2\ell$  by Corollary 2.3. Let  $\Phi = \{\varphi_j : 1 \leq j \leq 2\ell\}$  be a subset of  $B_\omega$  of cardinality  $2\ell$ . By Lemma 3.3 all the rows of the matrix  $J_{\Phi, t_o}$  vanish except possibly  $(\tau_{jh}\widehat{\varphi}_1, \tau_{jh}\widehat{\varphi}_2, \dots, \tau_{jh}\widehat{\varphi}_{2\ell})$ ,  $-\ell \leq j \leq \ell-1$ . Thus we identify the infinite matrices  $J_{\Phi, t_o}$ ,  $J_{\Phi, t_o}^*$  and  $G_{\Phi, t_o}$  with their  $2\ell \times 2\ell$  submatrices corresponding to these rows. The entries of the Gramian matrix are

$$(G_{\Phi, t_o})_{jk} = [\widehat{\varphi}_k, \widehat{\varphi}_j] \quad 1 \leq j \leq 2\ell \quad 1 \leq k \leq 2\ell.$$

By Lemma 3.3 the  $i$ -th column of  $J_{\Phi, t_o}$ ,  $1 \leq i \leq 2\ell$  is

$$(3.9) \quad \sqrt{h} \begin{bmatrix} 0 \\ \tau_{-(\ell-1)h}\widehat{\varphi}_i \\ \vdots \\ \widehat{\varphi}_i \\ \vdots \\ \tau_{(\ell-2)h}\widehat{\varphi}_i \\ \tau_{(\ell-1)h}\widehat{\varphi}_i \end{bmatrix} \text{ in } I_- \quad \sqrt{h} \begin{bmatrix} \tau_{-\ell h}\widehat{\varphi}_i \\ \tau_{-(\ell-1)h}\widehat{\varphi}_i \\ \vdots \\ \widehat{\varphi}_i \\ \vdots \\ \tau_{(\ell-2)h}\widehat{\varphi}_i \\ \tau_{(\ell-1)h}\widehat{\varphi}_i \end{bmatrix} \text{ in } I \quad \sqrt{h} \begin{bmatrix} \tau_{-\ell h}\widehat{\varphi}_i \\ \tau_{-(\ell-1)h}\widehat{\varphi}_i \\ \vdots \\ \widehat{\varphi}_i \\ \vdots \\ \tau_{(\ell-2)h}\widehat{\varphi}_i \\ 0 \end{bmatrix} \text{ in } I_+.$$

We note that

$$(3.10) \quad \text{rank } G_{\Phi, t_o} = \text{rank } J_{\Phi, t_o}.$$

We shall use the following result of Bownik [B, Thm 2.3] which characterizes the system of translates  $E_{\Phi, t_o}$  as being a frame or a Riesz family (for the space it generates) in terms of the “fibers”  $\{\mathcal{L}_h\widehat{\varphi}(x) : \varphi \in \Phi\}$ .

**Theorem 3.4.** *Suppose  $\Phi \subset L^2(\mathbb{R}^n)$  is countable and let  $H$  be the subspace of  $L^2(\mathbb{R})$  generated by  $E_{\Phi, t_o}$ . Then*

- (i)  $E_{\Phi, t_o}$  is a frame for  $H$  with constants  $A, B$  if and only if  $\{\mathcal{L}_h\widehat{\varphi}(x) : \varphi \in \Phi\}$  is a frame for  $\mathcal{R}_h(H)(x)$  with constants  $A, B$  for a.e.  $x \in [0, h]$ .
- (ii)  $E_{\Phi, t_o}$  is a Riesz basis for  $H$  with constants  $A, B$  if and only if  $\{\mathcal{L}_h\widehat{\varphi} : \varphi \in \Phi\}$  is a Riesz basis for  $\mathcal{R}_h(H)(x)$  with constants  $A, B$  for a.e.  $x \in [0, h]$ .

To apply Bownik’s theorem in our context we need a simple lemma of linear algebra. Let  $J$  be a  $n \times m$  matrix with complex entries,  $n \leq m$ ; we shall denote by  $\|J\|$  the norm of  $J$  as linear operator from  $\mathbb{C}^m$  to  $\mathbb{C}^n$  and by  $[J]_n$  the sum of the squares of the absolute values of the minors of order  $n$  of  $J$ .

**Lemma 3.5.** *Let  $v_1, \dots, v_m$  be  $m$  vectors in  $\mathbb{C}^n$ ,  $m \geq n$ , and denote by  $J$  the matrix  $(v_1, \dots, v_m)$  whose  $j$ -th column is the vector  $v_j$ .*

- (i) *If  $[J]_n > 0$  then  $\{v_1, \dots, v_m\}$  is a frame of  $\mathbb{C}^n$  with frame constants  $A \geq [J]_n \|J\|^{2(1-n)}$ ,  $B = \|J\|^2$ . Conversely, if  $\{v_1, \dots, v_m\}$  is a frame for  $\mathbb{C}^n$  with constants  $A$  and  $B$ , then  $[J]_n \geq A^n$  and  $\|J\| \leq B^{1/2}$ .*
- (ii) *If  $m = n$  and  $\det J > 0$  then  $\{v_1, \dots, v_m\}$  is a Riesz basis of  $\mathbb{C}^n$  with constants  $A = \det(J)^2 \|J\|^{2(1-n)}$  and  $B = \|J\|^2$ . Conversely, if  $\{v_1, \dots, v_m\}$  is a Riesz basis for  $\mathbb{C}^n$  with constants  $A$  and  $B$  then  $\det(J) \geq A^{n/2}$  and  $\|J\| \leq B^{1/2}$ .*

*Proof.* Let  $T : \mathbb{C}^m \rightarrow \mathbb{C}^n$  be the synthesis operator associated to  $\{v_1, \dots, v_m\}$ , i.e.  $Tz = \sum_{j=1}^m z_j v_j$  for all  $z \in \mathbb{C}^m$ . We observe that  $TT^*$  and  $JJ^*$  have the same eigenvalues  $\lambda_1 \leq \dots \leq \lambda_n$ , because the matrix  $J$  represents the operator  $T$  with respect to the canonical bases of  $\mathbb{C}^m$  and  $\mathbb{C}^n$ . Since

$$\lambda_1 I \leq TT^* \leq \lambda_n I$$

$\{v_1, \dots, v_m\}$  is a frame for  $\mathbb{C}^n$  if and only if  $\lambda_1 > 0$ . In such a case  $\lambda_1$  is the biggest lower frame bound and  $\lambda_n$  is the smallest upper frame bound. Moreover  $\lambda_n = \|JJ^*\| = \|J\|^2$ .

Now suppose that  $[J]_n > 0$ . By the Cauchy-Binet theorem  $[J]_n = \det(JJ^*) = \prod_{j=1}^n \lambda_j$ . Thus

$$\lambda_1 = \frac{\det(JJ^*)}{\prod_{k=2}^n \lambda_k} \geq \frac{[J]_n}{\lambda_n^{n-1}} = \frac{[J]_n}{\|J\|^{2(n-1)}} > 0$$

and  $\{v_1, \dots, v_m\}$  is a frame for  $\mathbb{C}^n$  with constants  $\lambda_1 \geq [J]_n \|J\|^{2(1-n)}$  and  $\lambda_n = \|J\|^2$ .

Conversely, suppose that  $\{v_1, \dots, v_m\}$  is a frame for  $\mathbb{C}^n$  with constants  $A$  and  $B$ . Then  $A \leq \lambda_1 \leq \lambda_n \leq B$ . Hence  $\|J\| \leq B^{1/2}$  and

$$[J]_n = \det(JJ^*) \geq \lambda_1^n \geq A^n > 0.$$

This concludes the proof of part (i) of the Lemma.

To prove the second part it suffices to observe that  $\{v_1, \dots, v_n\}$  is a Riesz basis of  $\mathbb{C}^n$  if and only if  $T$  is an isomorphism and that, in such a case, Riesz constants are also frame bounds. Moreover  $[J]_n = |\det J|^2$  when  $m = n$ .  $\square$

**Theorem 3.6.** *Suppose that  $\frac{\vartheta}{\ell} \leq h < \frac{\vartheta}{\ell - \frac{1}{2}}$ . Let  $\Phi = \{\varphi_j, 1 \leq j \leq 2\ell\}$  be a subset of  $B_\omega$ . Then  $E_{\Phi, t_o}$  is a frame for  $B_\omega$  if and only if there exist positive constants  $\delta, \gamma, \sigma$  and  $\eta$  such that*

$$(3.11) \quad \delta \leq \sum_{j=1}^{2\ell} |\widehat{\varphi}_j|^2 \leq \gamma \quad \text{a.e. in } (-\vartheta, \vartheta),$$

$$(3.12) \quad [J_{\Phi, t_o}]_{2\ell-1} \geq \sigma \quad \text{a.e. in } I_- \cup I_+,$$

$$(3.13) \quad |\det J_{\Phi, t_o}| \geq \eta \quad \text{a.e. in } I.$$

If  $h = \frac{\vartheta}{\ell}$  the intervals  $I_-$  and  $I_+$  are empty. In this case  $E_{\Phi, t_o}$  is a Riesz basis for  $B_\omega$  if and only if conditions (3.11) and (3.13) hold.

*Proof.* First we shall prove the theorem for  $\frac{\vartheta}{\ell} < h < \frac{\vartheta}{\ell - 1/2}$ .

Let  $E_{\Phi, t_o}$  be a frame for  $B_\omega$  with frame constants  $A$  and  $B$ . This implies in particular that  $B_\omega$  coincides with the space  $S_{\Phi, t_o}$  generated by  $E_{\Phi, t_o}$ . Condition (3.11) follows from Proposition 3.2. Thus we only need to prove (3.12) and (3.13).

We recall that the columns of the pre-Gramian  $J_{\Phi, t_o}$  are the vectors  $\mathcal{L}_h \widehat{\varphi}_j$ ,  $j = 1, \dots, 2\ell$  by (2.14). Thus, by Theorem 3.4(i), the columns of  $J_{\Phi, t_o}(x)$  are a frame with constants  $A, B$  for the space  $\mathcal{R}_h(B(\omega))(x)$  for a.e.  $x$ . By Theorem 2.2 we may identify canonically  $\mathcal{R}_h(B(\omega))(x)$  with  $\mathbb{C}^{2\ell-1}$  for a.e.  $x \in I_- \cup I_+$  and with  $\mathbb{C}^{2\ell}$  for a.e.  $x \in I$ . Thus, by applying Lemma 3.5(i) with  $v_j = \mathcal{L}_h \widehat{\varphi}_j(x)$  and  $J = J_{\Phi, t_o}(x)$ , we obtain that  $[J_{\Phi, t_o}(x)]_{2\ell-1} \geq A^{2\ell-1}$  for a.e.  $x \in I_- \cup I_+$  and  $\det(J_{\Phi, t_o}(x)) = [J_{\Phi, t_o}(x)]_{2\ell}^{1/2} \geq A^\ell$  for a.e.  $x \in I$ . This proves that conditions (3.11)-(3.13) are necessary.

To prove sufficiency assume that conditions (3.11)-(3.13) are satisfied. First we prove that the space  $S_{\Phi, t_o}$  spanned by  $E_{\Phi, t_o}$  is  $B_\omega$ . Since both are  $t_o$ -shift invariant spaces it is enough to show that their range functions coincide almost everywhere. We recall that the range  $\mathcal{R}_h(S_{\Phi, t_o})$  of  $S_{\Phi, t_o}$  is the space spanned by the columns of  $J_{\Phi, t_o}$ . If  $x$  is in  $I_-$  then by (3.9)  $\mathcal{R}_h(S_{\Phi, t_o})(x) \subseteq \text{span}\{\delta_j : |j| \leq \ell-1\}$  and the latter space coincides with  $\mathcal{R}_h(B_\omega)(x)$  by (2.22). On the other hand,  $\text{rank } J_{\Phi, t_o}(x) = 2\ell-1$  by (3.9) and assumption (3.12). Thus  $\mathcal{R}_h(S_{\Phi, t_o})(x) = \mathcal{R}_h(B_\omega)(x)$  because both have dimension  $2\ell-1$ . Similar arguments show that the range functions coincide almost everywhere also in  $I_+$  and in  $I$ .

Next we observe that  $\|J_{\Phi, t_o}(x)\| \leq \sqrt{2\ell\gamma}$  for a.e.  $x$  in  $[0, h]$  by (3.11). Moreover  $[J_{\Phi, t_o}(x)]_{2\ell-1} > 0$  for a.e.  $x$  in  $I_- \cup I_+$  by (3.12) and  $[J_{\Phi, t_o}(x)]_{2\ell} = |\det(J_{\Phi, t_o})|^2 > 0$  for a.e.  $x$  in  $I$  by (3.13). Thus, by Lemma 3.5(i), the family  $\{\mathcal{L}_h \widehat{\varphi}_1(x), \dots, \mathcal{L}_h \widehat{\varphi}_{2\ell}(x)\}$  is a frame for  $\mathcal{R}_h(B_\omega)(x)$  for a.e.  $x$  in  $[0, h]$ . The upper frame constant  $B = \|J_{\Phi, t_o}(x)\|^2$  is bounded from above by  $2\ell\gamma$  almost everywhere in  $[0, h]$ . The lower frame constant  $A$  is bounded from below by

$$[J_{\Phi, t_o}(x)]_{2\ell-1} \|J_{\Phi, t_o}(x)\|^{2(1-2\ell)} \geq \sigma(2\ell\gamma)^{(1-2\ell)} \quad \text{a.e. in } I_- \cup I_+$$

and by

$$|\det J_{\Phi, t_o}(x)|^2 \|J_{\Phi, t_o}(x)\|^{2(1-2\ell)} \geq \eta^2 (2\ell\gamma)^{(1-2\ell)} \quad \text{a.e. in } I.$$

Thus  $E_{\Phi, t_o}$  is a frame for  $B_\omega$  by Theorem 3.4(i). This concludes the proof of the theorem when  $\frac{\phi}{\ell} < h < \frac{\phi}{\ell-1/2}$ .

To prove that if  $h = \frac{\phi}{\ell}$  then  $E_{\Phi, t_o}$  is a Riesz basis for  $B_\omega$  if and only if conditions (3.11) and (3.13) hold, one argues in a similar way using Theorem 3.4(ii) and Lemma 3.5(ii). We omit the details.  $\square$

**Remark** If  $\ell = 1$  condition (3.12) is superfluous. Indeed, if  $h = \phi$  the intervals  $I_-$  and  $I_+$  are empty. If  $\phi < h < 2\phi$  then (3.12) follows from (3.11) because

$$[J_{\Phi, t_o}]_{2\ell-1} = \sum_{j=1}^2 [\widehat{\varphi}_j, \widehat{\varphi}_j] = \sum_k \tau_{kh} \sum_{j=1}^2 |\widehat{\varphi}_j|^2 = \begin{cases} \sum_{j=1}^2 |\widehat{\varphi}_j|^2 & \text{in } I_- \\ \tau_{-h} \sum_{j=1}^2 |\widehat{\varphi}_j|^2 & \text{in } I_+, \end{cases}$$

and the conclusion follows because  $I_- \subset (-\phi, \phi)$  and  $I_+ \subset \tau_h(-\phi, \phi)$ .

Next we consider the case  $\frac{\phi}{\ell-1/2} \leq h < \frac{\phi}{\ell-1}$ . Then  $\text{len}_{t_o}(B_\omega) = 2\ell-1$  by Corollary 2.3. Let  $\Phi = \{\varphi_j : 1 \leq j \leq 2\ell-1\}$  be a subset of  $B_\omega$  of cardinality  $2\ell-1$ . By Lemma 3.3 all the rows of the matrix  $J_{\Phi, t_o}$ , except possibly  $(\tau_{jh}\widehat{\varphi}_1, \tau_{jh}\widehat{\varphi}_2, \dots, \tau_{jh}\widehat{\varphi}_{2\ell})$ ,  $-\ell \leq j \leq \ell-1$  vanish. Thus we identify the infinite matrices  $J_{\Phi, t_o}$ ,  $J_{\Phi, t_o}^*$  and  $G_{\Phi, t_o}$  with their  $2\ell-1 \times 2\ell-1$  submatrices corresponding to these rows. The  $i$ -th column of  $J_{\Phi, t_o}$ ,  $1 \leq i \leq 2\ell-1$  is

$$(3.14) \quad \sqrt{h} \begin{bmatrix} \tau_{-(\ell-1)h} \widehat{\varphi}_i \\ \vdots \\ \widehat{\varphi}_i \\ \vdots \\ \tau_{(\ell-2)h} \widehat{\varphi}_i \\ \tau_{(\ell-1)h} \widehat{\varphi}_i \end{bmatrix} \text{ in } K_- \quad \sqrt{h} \begin{bmatrix} \tau_{-(\ell-1)h} \widehat{\varphi}_i \\ \vdots \\ \widehat{\varphi}_i \\ \vdots \\ \tau_{(\ell-2)h} \widehat{\varphi}_i \\ 0 \end{bmatrix} \text{ in } K \quad \sqrt{h} \begin{bmatrix} \tau_{-\ell h} \widehat{\varphi}_i \\ \tau_{-(\ell-1)h} \widehat{\varphi}_i \\ \vdots \\ \widehat{\varphi}_i \\ \vdots \\ \tau_{(\ell-2)h} \widehat{\varphi}_i \end{bmatrix} \text{ in } K_+.$$

**Theorem 3.7.** *Let  $\Phi \subset B_\omega$  such that  $\Phi = \{\varphi_j, 1 \leq j < 2\ell - 1\}$ , and  $\ell \neq 1$  such that  $\frac{\vartheta}{\ell-1} \leq h < \frac{\vartheta}{\ell-1}$ . Then  $E_{\Phi, t_o}$  is a frame for  $B_\omega$  if and only if there exist positive constants  $\delta, \gamma, \sigma$  and  $\eta$  such that*

$$(3.15) \quad \delta \leq \sum_{j=1}^{2\ell-1} |\widehat{\varphi}_j|^2 \leq \gamma \quad \text{a.e. in } (-\vartheta, \vartheta),$$

$$(3.16) \quad [J_{\Phi, t_o}]_{2\ell-2} \geq \sigma \quad \text{a.e. in } K,$$

$$(3.17) \quad |\det J_{\Phi, t_o}| \geq \eta \quad \text{a.e. in } K_- \cup K_+.$$

If  $h = \frac{\vartheta}{\ell-1}$  then  $E_{\Phi, t_o}$  is a Riesz basis if and only if conditions (3.15) and (3.17) hold.

The proof is similar to that of Theorem 3.6. We omit the details.

#### 4. THE DUAL GENERATORS

Let  $N = \text{len}_{t_o}(B_\omega)$  be the length of  $B_\omega$  as  $t_o$ -shift-invariant space and let  $\Phi = \{\varphi_1, \dots, \varphi_N\}$  be a subset of  $B_\omega$ . In this section we shall find the dual generators  $\Phi^*$  when  $E_{\Phi, t_o}$  is a Riesz basis or a frame of  $B_\omega$ . With a slight abuse of notation in this section we shall denote by  $\Phi$  the vector  $(\varphi_1, \dots, \varphi_N)$  and by  $\Phi^*$  the vector  $(\varphi_1^*, \dots, \varphi_N^*)$ .

It is well known that if  $E_{\Phi, t_o}$  is a Riesz basis for  $S_{\Phi, t_o}$  then the Gramian matrix is invertible and the Fourier transform of the dual generators are given by

$$(4.1) \quad \widehat{\Phi^*}^\top = \overline{G_{\Phi, t_o}^{-1}} \widehat{\Phi}^\top,$$

where  $v^\top$  denotes the transpose of the vector  $v$ . In Theorems 4.1 - 4.7 we give explicit formulas for the Fourier transforms of the dual generators when  $E_{\Phi, t_o}$  is a frame satisfying the hypothesis of Theorems 3.6 or 3.7. The proof is based on the dual Gramian matrix  $\widehat{G}_{\Phi, t_o}$  representation of the operator  $T_{\Phi, t_o} T_{\Phi, t_o}^*$ . From (3.4) we obtain

$$\mathcal{L}_h \mathcal{F} T_{\Phi, t_o} T_{\Phi, t_o}^* \mathcal{F}^{-1} \widehat{\varphi}_k^* = J_{\Phi, t_o} J_{\Phi, t_o}^* \mathcal{L}_h \widehat{\varphi}_k^* \quad k = 1, \dots, N.$$

By (2.4) the left hand side is equal to  $\mathcal{L}_h \widehat{\varphi}_k$ . Hence

$$\mathcal{L}_h \widehat{\varphi}_k = J_{\Phi, t_o} J_{\Phi, t_o}^* \mathcal{L}_h \widehat{\varphi}_k^*, \quad k = 1, \dots, N$$

which, by (2.14), can be written

$$(4.2) \quad J_{\Phi, t_o} = J_{\Phi, t_o} J_{\Phi, t_o}^* J_{\Phi^*, t_o}.$$

As in Section 2 we identify the infinite matrices  $J_{\Phi, t_o}$ ,  $J_{\Phi, t_o}^*$  and  $J_{\Phi^*, t_o}$  with  $N \times N$  matrices by neglecting their vanishing rows and columns (see the discussion after Lemma 3.3). Thus we shall interpret (4.2) as an identity between  $N \times N$  matrices.

Under the assumptions of Theorems 3.6 and 3.7 the interval  $[0, h]$  is the disjoint union of three intervals where the pre-Gramian is either invertible or has rank  $N-1$ . In the latter case either the first or the last row of the pre-Gramian vanishes. In this case we shall denote by  $\mathbb{J}_{\Phi, t_o}$  the  $(N-1) \times N$  submatrix of  $J_{\Phi, t_o}$  obtained by

deleting the vanishing row from  $J_{\Phi, t_o}$ . It is straightforward to see that in this case equation (4.2) reduces to

$$(4.3) \quad \mathbb{J}_{\Phi, t_o} = \mathbb{J}_{\Phi, t_o} \mathbb{J}_{\Phi, t_o}^* \mathbb{J}_{\Phi^*, t_o}.$$

We regard (4.2) and (4.3) as equations for the unknowns  $J_{\Phi^*, t_o}$  and  $\mathbb{J}_{\Phi^*, t_o}$ , respectively. In the intervals where the matrix  $J_{\Phi, t_o}$  is invertible we can solve for  $J_{\Phi^*, t_o}$  in (4.2), obtaining that

$$(4.4) \quad J_{\Phi^*, t_o} = (J_{\Phi, t_o}^*)^{-1}.$$

In the intervals where the rank of  $J_{\Phi, t_o}$  is  $N - 1$  we can solve for the submatrix  $\mathbb{J}_{\Phi^*, t_o}$  obtaining that

$$(4.5) \quad \mathbb{J}_{\Phi^*, t_o} = (\mathbb{J}_{\Phi, t_o} \mathbb{J}_{\Phi, t_o}^*)^{-1} \mathbb{J}_{\Phi, t_o}.$$

We recall that if  $A$  is a  $N \times (N - 1)$  matrix of rank  $N - 1$  then its Moore-Penrose inverse  $A^\dagger$  is

$$(4.6) \quad A^\dagger = (A^* A)^{-1} A^*$$

(see [BIG]). Therefore by (4.5)

$$(4.7) \quad \mathbb{J}_{\Phi^*, t_o} = (\mathbb{J}_{\Phi, t_o}^*)^\dagger.$$

We refer the reader to [BIG] for the definition and the properties of the Moore-Penrose inverse of a matrix.

By using (4.4) and (4.5) we shall obtain explicit formulas for the Fourier transforms of the dual generators. For the sake of clarity first we state and prove the result for  $N = 2, 3, 4$ . By Corollary 2.3 these cases correspond to  $\emptyset \leq h < 2\emptyset$ ,  $\frac{\emptyset}{2} \leq h < \frac{2}{3}\emptyset$  and  $\frac{2}{3}\emptyset \leq h < \emptyset$  respectively.

**Theorem 4.1.** *Assume that  $\emptyset \leq h < 2\emptyset$  and let  $\Phi$  denote the vector  $(\varphi_1, \varphi_2)$  where  $\varphi_1, \varphi_2 \in B_\omega$ . If (3.11) and (3.13) hold with  $\ell = 1$ , i.e. if  $E_{\Phi, t_o}$  is a frame for  $B_\omega$ , then*

$$(4.8) \quad \widehat{\Phi}^* = \begin{cases} D \tau_h \widehat{\Phi}^\perp & \text{in } [-\emptyset, \emptyset - h] \\ h^{-1} \|\widehat{\Phi}\|^{-2} \widehat{\Phi} & \text{in } (\emptyset - h, h - \emptyset) \\ -D \tau_{-h} \widehat{\Phi}^\perp & \text{in } [h - \emptyset, \emptyset] \end{cases}$$

where  $D = (\det J_{\Phi, t_o}^*)^{-1}$  and  $\widehat{\Phi}^\perp = (\widehat{\varphi}_2, -\widehat{\varphi}_1)$ . Note that if  $h = \emptyset$  the central interval is empty.

*Proof.* Assume first that  $\emptyset < h < 2\emptyset$ . We recall that  $I_- = (0, h - \emptyset)$ ,  $I = (h - \emptyset, \emptyset)$  and  $I_+ = (\emptyset, h)$ . The pre-Gramian  $J_{\Phi, t_o}$  is

$$\begin{aligned} & \sqrt{h} \begin{bmatrix} 0 & 0 \\ \widehat{\varphi}_1 & \widehat{\varphi}_2 \end{bmatrix} \text{ in } I_- & \sqrt{h} \begin{bmatrix} \tau_{-h} \widehat{\varphi}_1 & \tau_{-h} \widehat{\varphi}_2 \\ 0 & 0 \end{bmatrix} \text{ in } I_+ \\ & \sqrt{h} \begin{bmatrix} \tau_{-h} \widehat{\varphi}_1 & \tau_{-h} \widehat{\varphi}_2 \\ \widehat{\varphi}_1 & \widehat{\varphi}_2 \end{bmatrix} \text{ in } I. \end{aligned}$$

The same formulas hold for  $J_{\Phi^*, t_o}$  with  $\widehat{\varphi}_i$  replaced by  $\widehat{\varphi}_i^*$ ,  $j = 1, 2$ . Therefore

$$(4.9) \quad \mathbb{J}_{\Phi, t_o} = \begin{cases} \sqrt{h} \widehat{\Phi} & \text{in } I_- \\ \sqrt{h} \tau_{-h} \widehat{\Phi} & \text{in } I_+ \end{cases} \quad \mathbb{J}_{\Phi^*, t_o} = \begin{cases} \sqrt{h} \widehat{\Phi}^* & \text{in } I_- \\ \sqrt{h} \tau_{-h} \widehat{\Phi}^* & \text{in } I_+. \end{cases}$$

By assumptions (3.11) and (3.13)  $J_{\Phi, t_o}$  has rank 1 in  $I_- \cup I_+$  and rank 2 in  $I$ . Hence  $\mathbb{J}_{\Phi^*, t_o} = (\mathbb{J}_{\Phi, t_o}^*)^\dagger$  in  $I_- \cup I_+$  and

$$(4.10) \quad J_{\Phi^*, t_o} = J_{\Phi, t_o}^*{}^{-1} \quad \text{in } I.$$

First we find  $\mathbb{J}_{\Phi^*, t_o}$  in  $I_- \cup I_+$ . By using (4.5) we get

$$\mathbb{J}_{\Phi^*, t_o} = (\mathbb{J}_{\Phi, t_o}^*)^\dagger = \begin{cases} \frac{\widehat{\Phi}}{\sqrt{h\|\widehat{\Phi}\|^2}} & \text{in } I_- \\ \frac{\tau_{-h}\widehat{\Phi}}{\sqrt{h\|\tau_{-h}\widehat{\Phi}\|^2}} & \text{in } I_+. \end{cases}$$

By (4.9) we obtain that  $\widehat{\Phi}^* = \frac{\widehat{\Phi}}{h\|\widehat{\Phi}\|^2}$  in  $I_-$  and  $\tau_{-h}\widehat{\Phi}^* = \frac{\tau_{-h}\widehat{\Phi}}{h\|\tau_{-h}\widehat{\Phi}\|^2}$  in  $I_+$ . Since  $\tau_{-h}I_+ = (\emptyset - h, 0)$  we finally get

$$\widehat{\Phi}^* = \frac{\widehat{\Phi}}{h\|\widehat{\Phi}\|^2} \quad \text{in } (\emptyset - h, h - \emptyset).$$

Next we find the dual generators in the remaining intervals. By (4.10)

$$J_{\Phi^*, t_o} = \sqrt{h} (\det J_{\Phi, t_o}^*)^{-1} \begin{bmatrix} \widehat{\varphi_2} & -\widehat{\varphi_1} \\ -\tau_{-h}\widehat{\varphi_2} & \tau_{-h}\widehat{\varphi_1} \end{bmatrix} \quad \text{a.e. in } (h - \emptyset, \emptyset).$$

By translating and reminding that the pre-Gramian matrix is  $h$ -periodic

$$\widehat{\Phi}^* = \begin{cases} D\tau_h(\widehat{\varphi_2}, -\widehat{\varphi_1}) = D\tau_h\widehat{\Phi}^\perp & \text{in } (-\emptyset, \emptyset - h) \\ D\tau_{-h}(-\widehat{\varphi_2}, \widehat{\varphi_1}) = -D\tau_{-h}\widehat{\Phi}^\perp & \text{in } (h - \emptyset, \emptyset). \end{cases}$$

This completes the proof of the theorem when  $\emptyset < h < 2\emptyset$ .

If  $h = \emptyset$  one argues as before; the only difference is that now the interval  $(h - \emptyset, \emptyset - h)$  is empty.  $\square$

To find an explicit expression of the dual generators when  $N > 2$  we need formulas for the rows of the Moore-Penrose inverse of a  $N \times (N - 1)$  matrix of full rank. Given a  $(N - 1)$ -ple of vectors  $(U_1, \dots, U_{N-1})$  in  $\mathbb{C}^N$  their cross product is

$$\bigtimes_{j=1}^{N-1} U_j = U_1 \times U_2 \dots \times U_{N-1} = \det \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \dots & \mathbf{e}_N \\ U_1^1 & U_1^2 & \dots & U_1^N \\ \vdots & \vdots & & \vdots \\ U_{N-1}^1 & U_{N-1}^2 & \dots & U_{N-1}^N \end{bmatrix}$$

where the  $\{\mathbf{e}_j : j = 1, \dots, N\}$  is the canonical basis of  $\mathbb{C}^N$ . Notice that if  $N = 2$  then  $\bigtimes_{j=1}^{N-1} U = U^\perp$ . Given a vector  $W$  in  $\mathbb{C}^N$  and an integer  $k \in \{1, 2, \dots, N - 1\}$ ,

we shall denote by  $\bigtimes_{j=1}^{N-1} U_j \langle U_k \leftarrow W \rangle$  the cross product of the  $(N - 1)$ -ple  $(U_1, \dots, U_{k-1}, W, U_{k+1}, \dots, U_{N-1})$ , i.e.

$$\bigtimes_{j=1}^{N-1} U_j \langle U_k \leftarrow W \rangle = U_1 \times U_2 \dots \times U_{k-1} \times W \times U_{k+1} \dots \times U_{N-1}.$$



**Lemma 4.2.** *Let  $M$  be an  $n \times n$  invertible matrix. Denote by  $R_j$ ,  $j = 1, \dots, n$ , its rows and by  $C_j$  its columns. Then the columns of  $M^{-1}$  are*

$$(-1)^{k+1}(\det M)^{-1} \bigtimes_{\substack{j=1 \\ j \neq k}}^n R_j \quad 1 \leq k \leq n$$

and the rows are

$$(-1)^{k+1}(\det M)^{-1} \bigtimes_{\substack{j=1 \\ j \neq k}}^n C_j \quad 1 \leq k \leq n.$$

*Proof.* Let  $M_{kj}$  be the  $kj$ -cofactor of  $M$ . Then the  $k$ -th row of the matrix  $\text{cof}(M)$  of cofactors of  $M$  is

$$\begin{aligned} L_k &= (-1)^{k+1}(M_{k1}\mathbf{e}_1 - M_{k2}\mathbf{e}_2 + \dots + (-1)^{n-1}M_{kn}\mathbf{e}_n) \\ &= (-1)^{k+1} \bigtimes_{\substack{j=1 \\ j \neq k}}^n R_j. \end{aligned}$$

The first identity follows from the fact that  $M^{-1} = (\det M)^{-1} \text{cof}(M^\top)$ . The proof of the second identity is similar.  $\square$

**Lemma 4.3.** *Let  $A$  be a  $n \times (n-1)$  complex matrix of maximum rank. Denote by  $A_j$ ,  $j = 1, \dots, n-1$ , the columns of  $A$  and by  $P_k$ ,  $k = 1, \dots, n-1$  the rows of  $A^\dagger$ . Then*

$$(4.11) \quad P_k = (-1)^n [\det(A^* A)]^{-1} \bigtimes_{j=1}^{n-1} A_j \langle A_k \leftarrow W \rangle \quad 1 \leq k \leq n-1$$

where  $W = \bigtimes_{j=1}^{n-1} \bar{A}_j$ .

*Proof.* Since  $\text{rank } A = n-1$  the null space of  $A^*$  is the space spanned by  $W$ . Let  $A_b$  the matrix obtained by bordering  $A$  with the column  $W$ , i.e.

$$A_b = \begin{bmatrix} A & W \end{bmatrix}.$$

Then  $A_b$  is invertible because  $\det A_b = \bigtimes_{j=1}^{n-1} A_j \cdot W = |W|^2$ . By [BIG, Thm. 8]

$$A_b^{-1} = \begin{bmatrix} A^\dagger \\ W^\dagger \end{bmatrix}.$$

Thus, for every  $k = 1, \dots, n-1$ , the  $k$ -th row of  $A^\dagger$  is the  $k$ -th row of  $A_b^{-1}$ . Hence, by Lemma 4.2 and the anticommutativity of the cross product, we obtain that

$$\begin{aligned} P_k &= (-1)^{k+1} (\det A_b)^{-1} A_1 \times A_2 \times \dots \times A_{k-1} \times A_{k+1} \dots \times A_{n-1} \times W \\ &= (-1)^n |W|^{-2} \bigtimes_{j=1}^{n-1} A_j \langle A_k \leftarrow W \rangle. \end{aligned}$$

To conclude the proof we observe that  $|W|^2$  is the sum of the squares of the absolute values of the minors of order  $n-1$  of  $A$ , i.e. the determinant of  $A^*A$ , by the Cauchy-Binet formula.  $\square$

**Theorem 4.4.** *Assume that  $\frac{2}{3}\varnothing \leq h < \varnothing$  and let  $\Phi$  denote the vector  $(\varphi_1, \varphi_2, \varphi_3)$  where  $\varphi_j \in B_\omega$ ,  $j = 1, 2, 3$ . If assumptions (3.15)-(3.17) hold with  $\ell = 2$ , i.e. if  $E_{\Phi, t_o}$  is a frame for  $B_\omega$ , then the Fourier transform of the dual generators  $\Phi^* = (\varphi_1^*, \varphi_2^*, \varphi_3^*)$  is*

$$(4.12) \quad \widehat{\Phi}^* = \begin{cases} D \tau_h \widehat{\Phi} \times \tau_{2h} \widehat{\Phi} & \text{in } (-\varnothing, \varnothing - 2h) \\ E (\tau_h \widehat{\Phi} \times \widehat{\Phi}) \times \tau_h \widehat{\Phi} & \text{in } (\varnothing - 2h, h - \varnothing) \\ D \tau_h \widehat{\Phi} \times \tau_{-h} \widehat{\Phi} & \text{in } (h - \varnothing, \varnothing - h) \\ -E \tau_{-h} \widehat{\Phi} \times (\tau_{-h} \widehat{\Phi} \times \widehat{\Phi}) & \text{in } (\varnothing - h, 2h - \varnothing) \\ D \tau_{-2h} \widehat{\Phi} \times \tau_{-h} \widehat{\Phi} & \text{in } (2h - \varnothing, \varnothing) \end{cases}$$

where  $E = h (\det \mathbb{J}_{\Phi, t_o} \mathbb{J}_{\Phi, t_o}^*)^{-1}$  and  $D = \sqrt{h} (\det J_{\Phi, t_o}^*)^{-1}$ . Note that if  $h = \frac{2\varnothing}{3}$  the intervals  $(\varnothing - h, 2h - \varnothing)$   $(\varnothing - 2h, h - \varnothing)$  are empty.

*Proof.* Assume first that  $\frac{2\varnothing}{3} < h < \varnothing$ . We recall that  $K_-$ ,  $K$  and  $K_+$  denote the intervals  $(0, \varnothing - h)$ ,  $(\varnothing - h, 2h - \varnothing)$  and  $(2h - \varnothing, h)$  respectively, defined in (2.21). By (3.14) the matrix  $J_{\Phi, t_o}$  is

$$(4.13) \quad \sqrt{h} \begin{bmatrix} \tau_{-h} \widehat{\Phi} \\ \widehat{\Phi} \\ \tau_h \widehat{\Phi} \end{bmatrix} \text{ in } K_- \quad \sqrt{h} \begin{bmatrix} \tau_{-h} \widehat{\Phi} \\ \widehat{\Phi} \\ 0 \end{bmatrix} \text{ in } K \quad \sqrt{h} \begin{bmatrix} \tau_{-2h} \widehat{\Phi} \\ \tau_{-h} \widehat{\Phi} \\ \widehat{\Phi} \end{bmatrix} \text{ in } K_+.$$

The same formulas hold for  $J_{\Phi^*, t_o}$  with  $\widehat{\Phi}$  replaced by  $\widehat{\Phi}^*$ . Therefore

$$(4.14) \quad \mathbb{J}_{\Phi, t_o} = \sqrt{h} \begin{bmatrix} \tau_{-h} \widehat{\Phi} \\ \widehat{\Phi} \end{bmatrix} \quad \text{and} \quad \mathbb{J}_{\Phi^*, t_o} = \sqrt{h} \begin{bmatrix} \tau_{-h} \widehat{\Phi}^* \\ \widehat{\Phi}^* \end{bmatrix} \text{ in } K.$$

By assumptions (3.16) and (3.17) the matrix  $J_{\Phi, t_o}$  has rank 3 in  $K_- \cup K_+$  and rank 2 in  $K$ . Hence  $J_{\Phi^*, t_o} = J_{\Phi, t_o}^*{}^{-1}$  in  $K_- \cup K_+$  and  $\mathbb{J}_{\Phi^*, t_o} = (\mathbb{J}_{\Phi, t_o}^*)^\dagger$  in  $K$ .

First we find  $J_{\Phi^*, t_o}$  in  $K_-$ . By Lemma 4.2

$$\tau_{-h} \widehat{\Phi}^* = D \widehat{\Phi} \times \tau_h \widehat{\Phi}, \quad \widehat{\Phi}^* = D \tau_h \widehat{\Phi} \times \tau_{-h} \widehat{\Phi}, \quad \tau_h \widehat{\Phi}^* = D \tau_{-h} \widehat{\Phi} \times \widehat{\Phi} \quad \text{a.e. in } K_-$$

where  $D = \sqrt{h} (\det J_{\Phi, t_o}^*)^{-1}$ . By translating the first and the last identities and reminding that the pre-Gramian is  $h$ -periodic, we obtain

$$(4.15) \quad \widehat{\Phi}^* = \begin{cases} D \tau_h \widehat{\Phi} \times \tau_{2h} \widehat{\Phi} & \text{in } (-h, \varnothing - 2h) \\ D \tau_h \widehat{\Phi} \times \tau_{-h} \widehat{\Phi} & \text{in } (0, \varnothing - h) \\ D \tau_{-2h} \widehat{\Phi} \times \tau_{-h} \widehat{\Phi} & \text{in } (h, \varnothing). \end{cases}$$

The same calculation in  $K_+$  gives

$$\tau_{-2h} \widehat{\Phi}^* = D \tau_{-h} \widehat{\Phi} \times \widehat{\Phi} \quad \tau_{-h} \widehat{\Phi}^* = D \widehat{\Phi} \times \tau_{-2h} \widehat{\Phi} \quad \widehat{\Phi}^* = D \tau_{-2h} \widehat{\Phi} \times \tau_{-h} \widehat{\Phi}.$$

By translating the first two identities we obtain

$$(4.16) \quad \widehat{\Phi}^* = \begin{cases} D \quad \tau_h \widehat{\Phi} \times \tau_{2h} \widehat{\Phi} & \text{in } (-\varnothing, -h) \\ D \quad \tau_h \widehat{\Phi} \times \tau_{-h} \widehat{\Phi} & \text{in } (h - \varnothing, 0) \\ D \quad \tau_{-2h} \widehat{\Phi} \times \tau_{-h} \widehat{\Phi} & \text{in } (2h - \varnothing, h). \end{cases}$$

Next we find  $\mathbb{J}_{\Phi^*, t_o}$  in  $K$ . By (4.14) and Lemma 4.3 the rows of Moore-Penrose inverse of  $\mathbb{J}_{\Phi, t_o}^*$  are

$$-\sqrt{h}E (\tau_{-h} \widehat{\Phi} \times \widehat{\Phi}) \times \widehat{\Phi} \quad -\sqrt{h}E \tau_{-h} \widehat{\Phi} \times (\tau_{-h} \widehat{\Phi} \times \widehat{\Phi})$$

where  $E = h (\det \mathbb{J}_{\Phi, t_o} \mathbb{J}_{\Phi, t_o}^*)^{-1}$ . Hence, by (4.14),

$$\tau_{-h} \widehat{\Phi}^* = -E (\tau_{-h} \widehat{\Phi} \times \widehat{\Phi}) \times \widehat{\Phi} \quad \widehat{\Phi}^* = -E \tau_{-h} \widehat{\Phi} \times (\tau_{-h} \widehat{\Phi} \times \widehat{\Phi}).$$

By translating the first identity and using the anticommutativity of the cross product, we obtain

$$(4.17) \quad \widehat{\Phi}^* = \begin{cases} E \tau_h \widehat{\Phi} \times (\widehat{\Phi} \times \tau_h \widehat{\Phi}) & \text{in } (\varnothing - 2h, h - \varnothing) \\ E (\tau_{-h} \widehat{\Phi} \times \widehat{\Phi}) \times \tau_{-h} \widehat{\Phi} & \text{in } (\varnothing - h, 2h - \varnothing). \end{cases}$$

The conclusion follows from formulas (4.15), (4.16) and (4.17). This completes the proof of the theorem when  $\frac{2\varnothing}{3} < h < \varnothing$ .

If  $h = \frac{2\varnothing}{3}$  one argues as before; the only difference is that now the interval  $(\varnothing - h, 2h - \varnothing)$  is empty.  $\square$

**Theorem 4.5.** Assume that  $\frac{\varnothing}{2} \leq h < \frac{2}{3}\varnothing$  and let  $\Phi$  denote the vector  $(\varphi_1, \varphi_2, \varphi_3, \varphi_4)$  where  $\varphi_j \in B_\omega$ ,  $j = 1, \dots, 4$ . If (3.11)-(3.13) hold with  $\ell = 2$ , i.e. if  $E_{\Phi, t_o}$  is a frame for  $B_\omega$ , then the Fourier transform of the dual generators  $\Phi^* = (\varphi_1^*, \varphi_2^*, \varphi_3^*, \varphi_4^*)$  is

$$(4.18) \quad \widehat{\Phi}^* = \begin{cases} E \quad \tau_{-h} \widehat{\Phi} \times (\tau_{-h} \widehat{\Phi} \times \widehat{\Phi} \times \tau_h \widehat{\Phi}) \times \tau_h \widehat{\Phi} & \text{in } (0, 2h - \varnothing) \\ D \quad \tau_{-2h} \widehat{\Phi} \times \tau_{-h} \widehat{\Phi} \times \tau_h \widehat{\Phi} & \text{in } (2h - \varnothing, \varnothing - h) \\ E \quad \tau_{-2h} \widehat{\Phi} \times \tau_{-h} \widehat{\Phi} \times (\tau_{-2h} \widehat{\Phi} \times \tau_{-h} \widehat{\Phi} \times \widehat{\Phi}) & \text{in } (\varnothing - h, 3h - \varnothing) \\ D \quad \tau_{-3h} \widehat{\Phi} \times \tau_{-2h} \widehat{\Phi} \times \tau_{-h} \widehat{\Phi} & \text{in } (3h - \varnothing, \varnothing) \end{cases}$$

where  $E = h^2 (\det \mathbb{J}_{\Phi, t_o} \mathbb{J}_{\Phi, t_o}^*)^{-1}$  and  $D = h (\det \overline{J_{\Phi, t_o}})^{-1}$ . The expression of  $\Phi^*$  in  $(-\omega, 0)$  is obtained by reflecting each interval in (4.18) around zero and replacing  $\tau_{jh}$  with  $\tau_{-jh}$  in the expression of  $\widehat{\Phi}^*$  in the reflected interval. Note that if  $h = \frac{\varnothing}{2}$  then the first and third intervals are empty.

*Proof.* Assume first that  $\frac{\varnothing}{2} < h < \frac{2\varnothing}{3}$ . We recall that by  $I_-$ ,  $I$  and  $I_+$  we denote the intervals  $(0, 2h - \varnothing)$ ,  $(2h - \varnothing, \varnothing - h)$  and  $(\varnothing - h, h)$  defined in (2.20). By (3.9) the columns of  $J_{\Phi, t_o}$  are

$$\sqrt{h} \begin{bmatrix} 0 \\ \tau_{-h} \widehat{\Phi} \\ \widehat{\Phi} \\ \tau_h \widehat{\Phi} \end{bmatrix} \text{ in } I_- \quad \sqrt{h} \begin{bmatrix} \tau_{-2h} \widehat{\Phi} \\ \tau_{-h} \widehat{\Phi} \\ \widehat{\Phi} \\ \tau_h \widehat{\Phi} \end{bmatrix} \text{ in } I \quad \sqrt{h} \begin{bmatrix} \tau_{-2h} \widehat{\Phi} \\ \tau_{-h} \widehat{\Phi} \\ \widehat{\Phi} \\ 0 \end{bmatrix} \text{ in } I_+.$$

The same formulas hold for  $J_{\Phi^*, t_o}$  with  $\widehat{\Phi}$  replaced by  $\widehat{\Phi}^*$ . Therefore

$$(4.19) \quad \mathbb{J}_{\Phi, t_o} = \sqrt{h} \begin{bmatrix} \tau_{-h} \widehat{\Phi} \\ \widehat{\Phi} \\ \tau_h \widehat{\Phi} \end{bmatrix} \quad \text{and} \quad \mathbb{J}_{\Phi^*, t_o} = \sqrt{h} \begin{bmatrix} \tau_{-h} \widehat{\Phi}^* \\ \widehat{\Phi}^* \\ \tau_h \widehat{\Phi}^* \end{bmatrix} \quad \text{in } I_-$$

$$(4.20) \quad \mathbb{J}_{\Phi, t_o} = \sqrt{h} \begin{bmatrix} \tau_{-2h} \widehat{\Phi} \\ \tau_{-h} \widehat{\Phi} \\ \widehat{\Phi} \end{bmatrix} \quad \text{and} \quad \mathbb{J}_{\Phi^*, t_o} = \sqrt{h} \begin{bmatrix} \tau_{-2h} \widehat{\Phi}^* \\ \tau_{-h} \widehat{\Phi}^* \\ \widehat{\Phi}^* \end{bmatrix} \quad \text{in } I_+.$$

By assumptions (3.12) and (3.13) the matrix  $J_{\Phi, t_o}$  has rank 3 in  $I_- \cup I_+$  and rank 4 in  $I$ . Hence  $\mathbb{J}_{\Phi^*, t_o} = (\mathbb{J}_{\Phi, t_o}^*)^\dagger$  in  $I_- \cup I_+$  and  $J_{\Phi^*, t_o} = J_{\Phi, t_o}^*{}^{-1}$  in  $I$ .

First we find  $J_{\Phi^*, t_o}$  in  $I$ ; by Lemma 4.2

$$\begin{aligned} \tau_{-2h} \widehat{\Phi}^* &= D \quad \tau_{-h} \widehat{\Phi} \times \widehat{\Phi} \times \tau_h \widehat{\Phi} \\ \tau_{-h} \widehat{\Phi}^* &= -D \quad \tau_{-2h} \widehat{\Phi} \times \widehat{\Phi} \times \tau_h \widehat{\Phi} \\ \widehat{\Phi}^* &= D \quad \tau_{-2h} \widehat{\Phi} \times \tau_{-h} \widehat{\Phi} \times \tau_h \widehat{\Phi} \\ \tau_h \widehat{\Phi}^* &= -D \quad \tau_{-2h} \widehat{\Phi} \times \tau_{-h} \widehat{\Phi} \times \widehat{\Phi} \end{aligned}$$

in  $(2h - \varphi, \varphi - h)$ ; here  $D = h (\det \overline{J_{\Phi, t_o}})^{-1}$ . By translating and reminding that the matrix  $J_{\Phi, t_o}$  is  $h$ -periodic we obtain

$$(4.21) \quad \widehat{\Phi}^* = \begin{cases} D \quad \tau_h \widehat{\Phi} \times \tau_{2h} \widehat{\Phi} \times \tau_{3h} \widehat{\Phi} & \text{in } (-\varphi, \varphi - 3h) \\ -D \quad \tau_{-h} \widehat{\Phi} \times \tau_h \widehat{\Phi} \times \tau_{2h} \widehat{\Phi} & \text{in } (h - \varphi, \varphi - 2h) \\ D \quad \tau_{-2h} \widehat{\Phi} \times \tau_{-h} \widehat{\Phi} \times \tau_h \widehat{\Phi} & \text{in } (2h - \varphi, \varphi - h) \\ -D \quad \tau_{-3h} \widehat{\Phi} \times \tau_{-2h} \widehat{\Phi} \times \tau_{-h} \widehat{\Phi} & \text{in } (3h - \varphi, \varphi). \end{cases}$$

Notice that if  $[a, b]$  is any of the intervals in the r.h.s. of (4.21) the expression of  $\widehat{\Phi}^*$  in  $[a, b]$  can be obtained from that in  $[-b, -a]$ , by replacing  $h$  by  $-h$  in the translations  $\tau_{jh}$ ,  $|j| \leq 3$ .

Next we find the dual generators in the remaining intervals. First let us consider the interval  $I_-$ . Here by Lemma 4.3 the rows of the Moore-Penrose inverse of  $\mathbb{J}_{\Phi, t_o}^*$  are

$$\sqrt{h} E W \times \widehat{\Phi} \times \tau_h \widehat{\Phi}, \quad \sqrt{h} E \tau_{-h} \widehat{\Phi} \times W \times \tau_h \widehat{\Phi}, \quad \sqrt{h} E \tau_{-h} \widehat{\Phi} \times \widehat{\Phi} \times W$$

where  $W = \tau_{-h} \widehat{\Phi} \times \widehat{\Phi} \times \tau_h \widehat{\Phi}$  and  $E = h^2 (\det \mathbb{J}_{\Phi, t_o} \mathbb{J}_{\Phi, t_o}^*)^{-1}$ . By using (4.19) and translating we obtain

$$(4.22) \quad \widehat{\Phi}^* = \begin{cases} E \quad \tau_h W \times \tau_h \widehat{\Phi} \times \tau_{2h} \widehat{\Phi} & \text{in } (-h, h - \varphi) \\ E \quad \tau_{-h} \widehat{\Phi} \times W \times \tau_h \widehat{\Phi} & \text{in } (0, 2h - \varphi) \\ E \quad \tau_{-2h} \widehat{\Phi} \times \tau_{-h} \widehat{\Phi} \times \tau_{-h} W & \text{in } (h, 3h - \varphi). \end{cases}$$

Finally we consider the interval  $I_+$ . Here the rows of the Moore-Penrose inverse of  $\mathbb{J}_{\Phi, t_o}^*$  are

$$\sqrt{h} E W_o \times \tau_{-h} \widehat{\Phi} \times \widehat{\Phi}, \quad \sqrt{h} E \tau_{-2h} \widehat{\Phi} \times W_o \times \widehat{\Phi}, \quad \sqrt{h} E \tau_{-2h} \widehat{\Phi} \times \tau_{-h} \widehat{\Phi} \times W_o$$

where  $W_o = \tau_{-2h}\widehat{\Phi} \times \tau_{-h}\widehat{\Phi} \times \widehat{\Phi}$ . By using (4.20) and translating we obtain

$$\widehat{\Phi}^* = \begin{cases} E & \tau_{2h}W_o \times \tau_h\widehat{\Phi} \times \tau_{2h}\widehat{\Phi} & \text{in } (\varnothing - 3h, -h) \\ E & \tau_{-h}\widehat{\Phi} \times \tau_hW_o \times \tau_h\widehat{\Phi} & \text{in } (\varnothing - 2h, 0) \\ E & \tau_{-2h}\widehat{\Phi} \times \tau_{-h}\widehat{\Phi} \times W_o & \text{in } (\varnothing - h, h). \end{cases}$$

Since  $\tau_hW_o = W$  we obtain

$$(4.23) \quad \widehat{\Phi}^* = \begin{cases} E & \tau_hW \times \tau_h\widehat{\Phi} \times \tau_{2h}\widehat{\Phi} & \text{in } (\varnothing - 3h, -h) \\ E & \tau_{-h}\widehat{\Phi} \times W \times \tau_h\widehat{\Phi} & \text{in } (\varnothing - 2h, 0) \\ E & \tau_{-2h}\widehat{\Phi} \times \tau_{-h}\widehat{\Phi} \times \tau_{-h}W & \text{in } (\varnothing - h, h). \end{cases}$$

By comparing formulas (4.22) and (4.23) we see that the expressions of  $\widehat{\Phi}^*$  in intervals symmetric with respect to zero can be obtained from each other by replacing  $\tau_{jh}$  with  $\tau_{-jh}$  (note that replacing  $\tau_{jh}$  by  $\tau_{-jh}$  changes also the sign of  $W$ ).

Formulas (4.21), (4.22) and (4.23) give the dual generators. This completes the proof of the theorem when  $\frac{\varnothing}{2} < h < \frac{2\varnothing}{3}$ .

If  $h = \frac{\varnothing}{2}$  one argues as before; the only difference is that now the intervals  $(\varnothing - 2h, 2h - \varnothing)$ ,  $(-h, h - \varnothing)$  and  $(\varnothing - h, h)$  are empty.  $\square$

Theorems 4.6 and 4.7 below generalize Theorems 4.4 and 4.5 respectively. We omit the proofs, which are analogous to the proofs of Theorems 4.4 and 4.5. We recall that  $K_-$ ,  $K$ , and  $K_+$  denote the intervals  $(0, \varnothing - (\ell - 1)h)$ ,  $(\varnothing - (\ell - 1)h, -\varnothing + \ell h)$ , and  $(-\varnothing + \ell h, h)$  respectively.

**Theorem 4.6.** *Let  $\frac{\varnothing}{\ell - \frac{1}{2}} \leq h < \frac{\varnothing}{\ell - 1}$  and denote by  $\Phi$  the vector  $(\varphi_1, \varphi_2, \dots, \varphi_{2\ell - 1})$ , where  $\varphi_j \in B_\omega$ ,  $j = 1, \dots, 2\ell - 1$ . If assumptions (3.15)-(3.17) hold, i.e. if  $E_{\Phi, t_o}$  is a frame for  $B_\omega$ , then the Fourier transform of the dual generators  $\Phi^* = (\varphi_1^*, \varphi_2^*, \dots, \varphi_{2\ell - 1}^*)$  is*

$$\widehat{\Phi}^* = \begin{cases} (-1)^{\ell+1-k} D \prod_{\substack{j=-\ell-k+1 \\ j \neq 0}}^{\ell-k-1} \tau_{jh}\widehat{\Phi} & \text{in } \tau_{kh}K_- \text{ for } -(\ell - 1) \leq k \leq \ell - 1 \\ -E \prod_{j=-\ell-k+1}^{\ell-k-2} \tau_{jh}\widehat{\Phi} \langle \widehat{\Phi} \leftarrow \tau_{-kh}W \rangle & \text{in } \tau_{kh}K \text{ for } -(\ell - 1) \leq k \leq \ell - 2 \\ (-1)^{\ell+k} D \prod_{\substack{j=-\ell-k \\ j \neq 0}}^{\ell-2-k} \tau_{jh}\widehat{\Phi} & \text{in } \tau_{kh}K_+ \text{ for } -\ell \leq k \leq \ell - 2 \end{cases}$$

where  $W = \prod_{j=-\ell+1}^{\ell-2} \tau_{jh}\widehat{\Phi}$ ,  $E = h^{2\ell-3} (\det \mathbb{J}_{\Phi, t_o} \mathbb{J}_{\Phi, t_o}^*)^{-1}$  and  $D = h^{\ell-\frac{3}{2}} (\det J_{\Phi, t_o}^*)^{-1}$ .

We recall that  $I_-$ ,  $I$ ,  $I_+$  denote the intervals  $(0, -\varnothing + \ell h)$ ,  $(-\varnothing + \ell h, \varnothing - (\ell - 1)h)$ , and  $(\varnothing - (\ell - 1)h, h)$  (see (2.20)).

**Theorem 4.7.** *Let  $\frac{\varnothing}{\ell} \leq h < \frac{\varnothing}{\ell - \frac{1}{2}}$  and denote by  $\Phi$  the vector  $(\varphi_1, \varphi_2, \dots, \varphi_{2\ell})$  where  $\varphi_j \in B_\omega$ ,  $j = 1, \dots, 2\ell$ . If assumptions (3.11)-(3.13) hold, i.e. if  $E_{\Phi, t_o}$  is a frame*

for  $B_\omega$ , then the Fourier transform of the dual generators  $\Phi^* = (\varphi_1^*, \varphi_2^*, \dots, \varphi_{2\ell}^*)$  is

$$\widehat{\Phi}^* = \begin{cases} E \prod_{j=-\ell-k+1}^{\ell-k-1} \tau_{jh} \widehat{\Phi} \langle \widehat{\Phi} \leftarrow \tau_{-kh} W \rangle & \text{in } \tau_{kh} I_- \quad \text{for } -(\ell-1) \leq k \leq \ell-1 \\ (-1)^{\ell+k} D \prod_{\substack{j=-\ell-k \\ j \neq 0}}^{\ell-k-1} \tau_{jh} \widehat{\Phi} & \text{in } \tau_{kh} I \quad \text{for } -\ell \leq k \leq \ell-1 \\ E \prod_{j=-\ell-k}^{\ell-k-2} \tau_{jh} \widehat{\Phi} \langle \widehat{\Phi} \leftarrow \tau_{-kh} W_o \rangle & \text{in } \tau_{kh} I_+ \quad \text{for } -\ell \leq k \leq \ell-2 \end{cases}$$

where  $W = \prod_{j=-\ell+1}^{\ell-1} \tau_{jh} \widehat{\Phi}$ ,  $W_o = \prod_{j=-\ell}^{\ell-2} \tau_{jh} \widehat{\Phi}$ ,  $E = h^{2\ell-2} (\det \mathbb{J}_{\Phi, t_o} \mathbb{J}_{\Phi, t_o}^*)^{-1}$  and  $D = h^{\ell-1} (\det J_{\Phi, t_o}^*)^{-1}$ .

## 5. SAMPLING FORMULAS FOR THE SPACE $B_\omega$

In this section we shall apply the previous results to oversampling formulas for the Hilbert transform sampling and the derivative sampling in  $B_\omega$ . In the derivative sampling formula the coefficients are the values of the function and of its derivatives  $f^{(j)}$ ,  $1 \leq j \leq K$ , at the points of a uniform grid on  $\mathbb{R}$ . It was first obtained by D. Jagerman and L. Fogel for  $K = 1$  and by Linden and N. M. Abramson for any  $K$  [JF] [L] [LA]. Successively J. R. Higgins derived the same expansion formulas by using the Riesz basis method [Hi].

In [SF] D.M.S. Santos and P.J.S.G. Ferreira have obtained a two-channel derivative oversampling formula for  $B_{\omega_a}$  with  $\omega_a < \omega$  by projecting both the Riesz basis generators of the space  $B_\omega$  and their duals into the space  $B_{\omega_a}$ . With this technique the projected family is a frame; however notice that projecting the dual of a Riesz basis does not yield the dual frame. Thus the coefficients of the expansions of a function computed with respect to the projected duals are not minimal in least square norm.

Let  $t_o$  be such that  $\varnothing \leq h < 2\varnothing$  and let  $\Phi = (\varphi_1, \varphi_2)$  be a vector such that  $E_{\Phi, t_o}$  is a frame for  $B_\omega$ . Then by (2.5)

$$(5.1) \quad f = \sum_{i=1,2} \sum_{k \in \mathbb{Z}} \langle f, \tau_{kt_o} \varphi_i \rangle \tau_{kt_o} \varphi_i^* \quad \forall f \in B_\omega.$$

By using the Plancherel and the inversion formulas we see that the coefficients

$$(5.2) \quad \langle f, \tau_{kt_o} \varphi_i \rangle = (\mathcal{M}_j f)(kt_o) \quad j = 1, 2$$

are the samples of the functions  $\mathcal{M}_j f = \mathcal{F}^{-1} \widehat{\varphi}_j \mathcal{F} f$  at the points  $kt_o$ ,  $k \in \mathbb{Z}$ . For this reason (5.1) is called a *sampling formula*. These formulas are useful in applications when one wants to reconstruct a signal from samples taken from two transformed version of the signal. For instance one may want to reconstruct  $f$  from samples of  $f$  and  $f'$  (derivative sampling) or from samples of  $f$  and its Hilbert transform  $\mathcal{H}f = -i \mathcal{F}^{-1} \text{sign} \mathcal{F} f$  (Hilbert transform sampling). Both are particular cases of the family of frames generated by the translates of two functions  $\varphi_1, \varphi_2$  such that  $\widehat{\varphi}_1 = \chi_{[-\varnothing, \varnothing]}$ ,  $\widehat{\varphi}_2 = m \chi_{[-\varnothing, \varnothing]}$ , where  $m$  is a function in  $L^\infty(\mathbb{R})$ .

**Proposition 5.1.** *Let  $m$  be a function in  $L^\infty(\mathbb{R})$  and let  $\Phi = (\varphi_1, \varphi_2)$  where*

$$(5.3) \quad \widehat{\varphi}_1 = \chi_{[-\varnothing, \varnothing]} \quad \widehat{\varphi}_2(x) = m\chi_{[-\varnothing, \varnothing]}.$$

*Suppose that  $\varnothing \leq h < 2\varnothing$ . Then  $E_{\Phi, t_\varnothing}$  is a frame for  $B_\omega$  if and only if there exists a positive number  $\eta$  such that*

$$(5.4) \quad |m - \tau_{-h}m| \geq \eta \quad \text{a.e. in } (h - \varnothing, \varnothing).$$

*The Fourier transforms of the dual generators are*

$$\widehat{\varphi}_1^* = \begin{cases} \frac{\tau_h \overline{m}}{h(\tau_h \overline{m} - \overline{m})}, & \text{in } [-\varnothing, \varnothing - h] \\ \frac{1}{h(1 + |m|^2)}, & \text{in } (\varnothing - h, h - \varnothing) \\ \frac{-\tau_{-h} \overline{m}}{h(\overline{m} - \tau_{-h} \overline{m})}, & \text{in } [h - \varnothing, \varnothing] \end{cases} \quad \widehat{\varphi}_2^* = \begin{cases} \frac{-1}{h(\tau_h \overline{m} - \overline{m})}, & \text{in } [-\varnothing, \varnothing - h] \\ \frac{m}{h(1 + |m|^2)}, & \text{in } (\varnothing - h, h - \varnothing) \\ \frac{1}{h(\overline{m} - \tau_{-h} \overline{m})}, & \text{in } [h - \varnothing, \varnothing]. \end{cases}$$

*If  $h = \varnothing$  then  $E_{\Phi, t_\varnothing}$  is a Riesz basis for  $B_\omega$ .*

*Proof.* Since  $\det J_{\Phi, t_\varnothing}^* = h(\overline{m} - \tau_{-h} \overline{m})$  the assumptions of Theorem 3.6 are satisfied. The expression of the Fourier transforms of the dual generators can be easily obtained from Theorem 4.1.  $\square$

By choosing  $m(x) = -i \operatorname{sign}(x)$  in (5.3) we obtain the Hilbert transform frames for  $B_\omega$ . For  $h = \varnothing$  the associated sampling formula is known as the *Hilbert transform sampling formula* (see [Hi, Ex.12.9]). The coefficients of the expansion are the values of the function  $f$  and its Hilbert transform  $\mathcal{H}f$  at the sample points  $kt_\varnothing, k \in \mathbb{Z}$ . Denote by  $\operatorname{sinc}$  the function  $\sin(x)/x$ .

**Corollary 5.2.** *Let  $\varphi_1, \varphi_2$  be defined by*

$$(5.5) \quad \widehat{\varphi}_1 = \chi_{[-\varnothing, \varnothing]} \quad \widehat{\varphi}_2 = -i\chi_{[-\varnothing, \varnothing]}\operatorname{sign}.$$

*If  $\varnothing \leq h < 2\varnothing$  then  $E_{\Phi, t_\varnothing}$  is a tight frame for  $B_\omega$ . The dual generators are  $\varphi_i^* = (2h)^{-1}\varphi_i$  for  $i = 1, 2$ . If  $h = \varnothing$  then  $E_{\Phi, t_\varnothing}$  is a Riesz basis for  $B_\omega$ . Moreover for any  $f \in B_\omega$  the following Hilbert transform sampling formula holds*

$$f(x) = \frac{\varnothing}{h} \sum_{k \in \mathbb{Z}} \left( f(kt_\varnothing) \tau_{-kt_\varnothing} \cos\left(\frac{\varnothing x}{2}\right) \operatorname{sinc}\left(\frac{\varnothing x}{2}\right) - (\mathcal{H}f)(kt_\varnothing) \tau_{-kt_\varnothing} \sin\left(\frac{\varnothing x}{2}\right) \operatorname{sinc}\left(\frac{\varnothing x}{2}\right) \right).$$

*Proof.* The assumptions of Proposition 5.1 are satisfied. Thus  $E_{\Phi, t_\varnothing}$  is a frame and the expression of the duals follows immediately. The frame is tight because  $TT^* = \frac{1}{2h}I$ , since  $\Phi^* = \frac{1}{2h}\Phi$ .

Standard calculations show that if  $\varphi_1$  and  $\varphi_2$  are the functions given by (5.5) then

$$(5.6) \quad \varphi_1(x) = \frac{\sqrt{2}}{\sqrt{\pi}} \varnothing \cos\left(\frac{\varnothing x}{2}\right) \operatorname{sinc}\left(\frac{\varnothing x}{2}\right) \quad \varphi_2(x) = \frac{\sqrt{2}}{\sqrt{\pi}} \varnothing \sin\left(\frac{\varnothing x}{2}\right) \operatorname{sinc}\left(\frac{\varnothing x}{2}\right).$$

Moreover the coefficients of the expansion formula (5.1) are  $\sqrt{2\pi}\hat{f}(kt_\varnothing)$  and  $-\sqrt{2\pi}(\mathcal{H}f)(kt_\varnothing)$ ,  $k \in \mathbb{Z}$ . This proves also the expansion formula.  $\square$

By choosing  $m(x) = ix$  in (5.5) we obtain the derivative frame for  $B_\omega$ . Given a function  $g$  we shall denote by  $g_\delta$  the function  $g(\delta x)$ . Note that  $\delta \operatorname{sinc}_\delta = \sqrt{\frac{\pi}{2}} \widehat{\chi}_{[-\delta, \delta]}$ .

**Corollary 5.3.** *Let  $\varphi_1, \varphi_2$  be defined by*

$$(5.7) \quad \widehat{\varphi}_1 = \chi_{[-\varnothing, \varnothing]} \quad \widehat{\varphi}_2 = ix\chi_{[-\varnothing, \varnothing]}.$$

*If  $\varnothing \leq h < 2\varnothing$  then  $E_{\Phi, t_o}$  is a frame for  $B_\omega$ ; if  $h = \varnothing$  then it is a Riesz basis for  $B_\omega$ . The Fourier transforms of the dual generators are*

$$\widehat{\varphi}_1^*(x) = \begin{cases} \frac{1}{h}(1 - \frac{|x|}{h}) & h - \varnothing < |x| < \varnothing \\ \frac{1}{h(1+x^2)} & |x| < h - \varnothing \end{cases} \quad \widehat{\varphi}_2^*(x) = \begin{cases} \frac{i}{h^2} \text{sign}(x) & h - \varnothing < |x| < \varnothing \\ \frac{ix}{h(1+x^2)} & |x| < h - \varnothing. \end{cases}$$

*Moreover for any  $f \in B_\omega$  the following derivative sampling formula holds*

$$(5.8) \quad f = \sqrt{2\pi} \sum_{k \in \mathbb{Z}} \left( f(kt_o) \tau_{-kt_o} \varphi_1^* - f'(kt_o) \tau_{-kt_o} \varphi_2^* \right),$$

where

$$\varphi_1^*(x) = e^{-|\cdot|} * (h - \varnothing) \frac{1}{h} \text{sinc}_{h-\varnothing}(x) + \frac{1}{\pi} (\varnothing \text{sinc}_\varnothing - (h - \varnothing) \text{sinc}_{h-\varnothing}) * \text{sinc}_{h/2}^2(x)$$

$$\varphi_2^*(x) = e^{-|\cdot|} * (h - \varnothing) \frac{1}{h} \text{sinc}'_{h-\varnothing}(x) + \frac{\sqrt{2}}{\sqrt{\pi}} \frac{1}{xh^2} (\cos_\varnothing - \cos_{h-\varnothing})(x).$$

*Proof.* By Proposition 5.1  $E_{\Phi, t_o}$  is a frame for all  $\varnothing \leq h < 2\varnothing$  and it is a Riesz basis if  $h = \varnothing$ . From (5.2) we see that the coefficients of the expansion (5.1) are  $\sqrt{2\pi} \hat{f}(kt_o)$  and  $-\sqrt{2\pi} f'(kt_o)$ . The expression of the dual generators can be obtained from  $\widehat{\varphi}_1^*$  and  $\widehat{\varphi}_2^*$  by computing the inverse Fourier transform.  $\square$

A simple calculation shows that if  $h = \varnothing$  then

$$\varphi_1^*(x) = \frac{1}{\sqrt{2\pi}} \text{sinc}^2\left(\frac{\varnothing x}{2}\right) \quad \varphi_2^*(x) = -\frac{1}{\sqrt{2\pi}} x \text{sinc}^2\left(\frac{\varnothing x}{2}\right).$$

Figure 1 and Figure 2 below show the Fourier transforms of the dual generators in the Riesz basis case  $h = \varnothing = 1$ , and in the case  $\varnothing = 1$  and  $h = \frac{3}{2}\varnothing$  respectively.

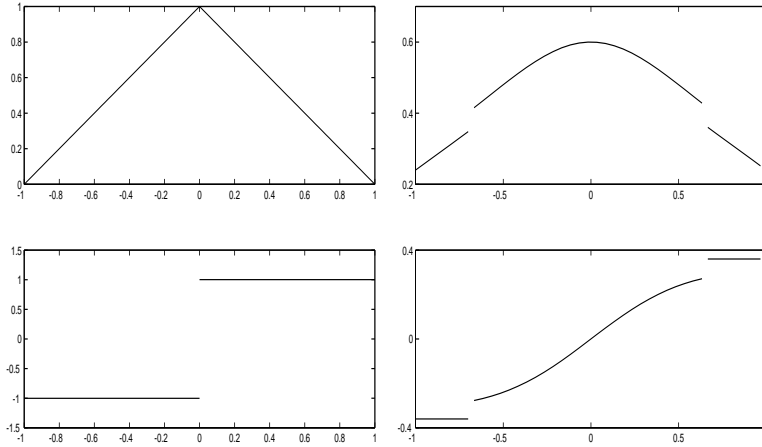


Figure 1:  $\widehat{\varphi}_1^*, \frac{1}{i}\widehat{\varphi}_2^*$  for  $h = \varnothing = 1$

Figure 2:  $\widehat{\varphi}_1^*, \frac{1}{i}\widehat{\varphi}_2^*$  for  $h = \frac{3}{2}\varnothing = \frac{3}{2}$



**Remark** In [JF] Jagerman and Fogel proved the following two-channel derivative sampling formula in the case  $h = \varnothing$  i.e.  $t_o = \frac{2\pi}{\varnothing}$

$$f(x) = \sum_k f(kt_o) \left( \operatorname{sinc}^2 \frac{\varnothing}{2} (x - kt_o) + \frac{2}{\varnothing} f'(kt_o) \sin \frac{\varnothing}{2} (x - kt_o) \operatorname{sinc} \frac{\varnothing}{2} (x - kt_o) \right)$$

(see also [Hi, p.135]). Thus formula (5.8) is an extension of the case  $t_o = \frac{2\pi}{\varnothing}$  to all values of  $t_o \in [\frac{\pi}{\varnothing}, \frac{2\pi}{\varnothing}]$  (i.e. for all  $h$  such that  $\varnothing \leq h < 2\varnothing$ ).

Our last example is a three channel derivative oversampling formula. To obtain a frame with three generators we must choose  $\frac{2}{3}\varnothing \leq h < \varnothing$  i.e.  $t_o \in [\frac{3\pi}{\varnothing}, \frac{2\pi}{\varnothing}]$ .

**Corollary 5.4.** Let  $\widehat{\varphi}_1, \widehat{\varphi}_2, \widehat{\varphi}_3$  be defined by

$$\widehat{\varphi}_1 = \chi_{[-\varnothing, \varnothing]}, \quad \widehat{\varphi}_2 = ix\chi_{[-\varnothing, \varnothing]}, \quad \widehat{\varphi}_3 = -x^2\chi_{[-\varnothing, \varnothing]}.$$

If  $\frac{2\varnothing}{3} \leq h < \varnothing$  then  $E_{\Phi, t_o}$  is a frame for  $B_\omega$ ; if  $h = \frac{2\varnothing}{3}$  then it is a Riesz basis for  $B_\omega$ . The Fourier transform of the dual generators are

$$(5.9) \quad \widehat{\Phi}^*(x) = \begin{cases} \frac{1}{2h^3} (x^2 + 3hx + 2h^2, -i(2x + 3h), -1) & -\varnothing < x < \varnothing - 2h \\ (A_h(x), B_h(x), C_h(x)) & \varnothing - 2h < x < h - \varnothing \\ \frac{1}{h^3} (h^2 - x^2, 2ix, 1) & h - \varnothing < x < \varnothing - h \\ (A_{-h}(x), B_{-h}(x), C_{-h}(x)) & \varnothing - h < x < 2h - \varnothing \\ \frac{1}{2h^3} (x^2 - 3hx + 2h^2, -i(2x - 3h), -1) & 2h - \varnothing < x < \varnothing \end{cases}$$

where

$$A_h(x) = \frac{1}{h^2} \frac{(x+h) + (2x+h)(x+h)^2}{1 + (2x+h)^2 + x^2(x+h)^2} \quad B_h(x) = \frac{-i}{h^2} \frac{1 - x(x+h)^3}{1 + (2x+h)^2 + x^2(x+h)^2}$$

$$C_h(x) = \frac{1}{h^2} \frac{h + 2x + x(x+h)^2}{1 + (2x+h)^2 + x^2(x+h)^2}.$$

Note that if  $h = \frac{2}{3}\varnothing$  the intervals  $(\varnothing - h, 2h - \varnothing)$  and  $(2h - \varnothing, h - \varnothing)$  are empty.

*Proof.* The family  $E_{\Phi, t_o}$  is a frame for  $B_\omega$  by Theorem 4.4. To compute the dual generators we used formula (4.12).  $\square$

Thus for each value of the parameter  $h$  in  $[\frac{2}{3}\varnothing, \varnothing)$  the family  $E_{\Phi, t_o}$  is a frame and every signal in  $B_\omega$  can be reconstructed from the values  $f(kt_o), f^{(1)}(kt_o), f^{(2)}(kt_o), k \in \mathbb{Z}$ , by the following three-channel derivative sampling formula

$$(5.10) \quad f = \frac{1}{\sqrt{2\pi}} \sum_{i=1}^3 \sum_{k \in \mathbb{Z}} (-1)^{i-1} f^{(i-1)}(kt_o) \varphi_i^*(x - kt_o),$$

where the dual generators are given by (5.9).

In the rest of this section we present two numerical experiments of reconstruction of a band limited signal by using formula (5.10). We have chosen the signal  $f(x) = \frac{1}{\sqrt{2\pi}} (\operatorname{sinc}(x/2))^2$  (see Figure 3); note that the function  $f$  is the Fourier transform of the function  $(1 - |x|)_+$ , therefore  $\varnothing = 1$ .

In the first experiment we take  $h = \frac{2}{3}\varnothing$  so that  $E_{\Phi, t_o}$  is a Riesz basis; in the second experiment we take  $h = \frac{11}{15}\varnothing$  so that  $E_{\Phi, t_o}$  is a frame. We observe that in the first case it is possible to find the analitic expression of the functions  $\varphi_j^*$  in (5.9),

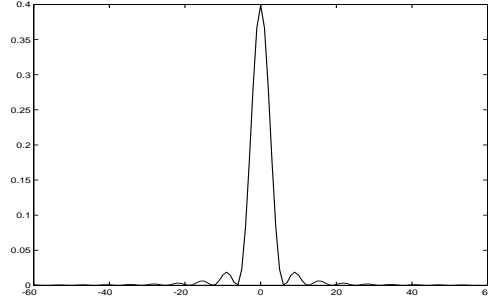


Figure 3: Signal  $f(x) = \frac{1}{\sqrt{2\pi}}(\text{sinc}(x/2))^2$

while in the second case they must be computed numerically. Indeed for  $h = \frac{2\phi}{3}$  one obtains

$$\begin{aligned}\varphi_1^* &= \frac{1}{\sqrt{2\pi}} \left(1 + \frac{\phi^2 x^2}{18}\right) \text{sinc}^3\left(\frac{\phi}{3}x\right) \\ \varphi_2^* &= -\frac{1}{\sqrt{2\pi}} x \text{sinc}^3\left(\frac{\phi}{3}x\right) \\ \varphi_3^* &= \frac{1}{2\sqrt{2\pi}} x^2 \text{sinc}^3\left(\frac{\phi}{3}x\right).\end{aligned}$$

Figures 4 and 5 below show the functions  $\varphi_1^*, \varphi_2^*, \varphi_3^*$  and  $\widehat{\varphi}_1^*, \frac{1}{i}\widehat{\varphi}_2^*, \widehat{\varphi}_3^*$  for  $h = \frac{2}{3}$ .

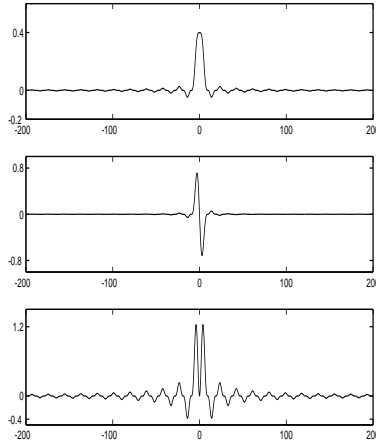


Figure 4:  $\Phi^*$ ;  $h = \frac{2}{3}\phi$

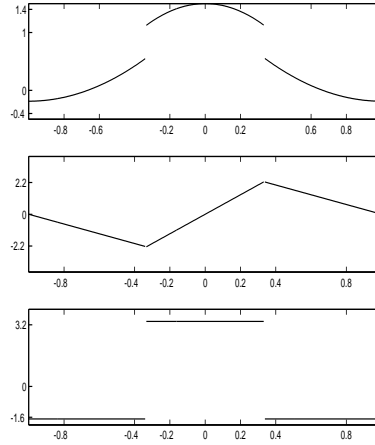


Figure 5:  $\widehat{\varphi}_1^*, \frac{1}{i}\widehat{\varphi}_2^*, \widehat{\varphi}_3^*$ ;  $h = \frac{2}{3}\phi$

Since in the case  $h = \frac{2}{3}\phi$  the expression of the duals is known, it is possible to write explicitly the sampling formula (5.10):

$$f(x) = \frac{1}{\sqrt{2\pi}} \sin^3\left(\frac{\phi x}{3}\right) \sum_n (-1)^n \left[ f\left(\frac{3n\pi}{\phi}\right) \frac{1}{\left(x - \frac{3n\pi}{\phi}\right)^3} + \right. \\ \left. + f\left(\frac{3n\pi}{\phi}\right) \frac{\phi^2/9}{\left(x - \frac{3n\pi}{\phi}\right)} - f^{(1)}\left(\frac{3n\pi}{\phi}\right) \frac{1}{\left(x - \frac{3n\pi}{\phi}\right)^2} + f^{(2)}\left(\frac{3n\pi}{\phi}\right) \frac{1}{2\left(x - \frac{3n\pi}{\phi}\right)^2} \right].$$

This formula was first given by Linden in [L] and Linden and Abramson in [LA], see also [HS].

In the case  $h = \frac{11}{12}\phi$  the dual generators have been obtained by computing numerically the inverse Fourier transforms of the functions in (5.9): Figure 6 and Figure 7 below show the functions  $\varphi_1^*, \varphi_2^*, \varphi_3^*$  and  $\widehat{\varphi}_1^*, \frac{1}{i}\widehat{\varphi}_2^*, \widehat{\varphi}_3^*$ ;  $h = \frac{11}{15}\phi$ .

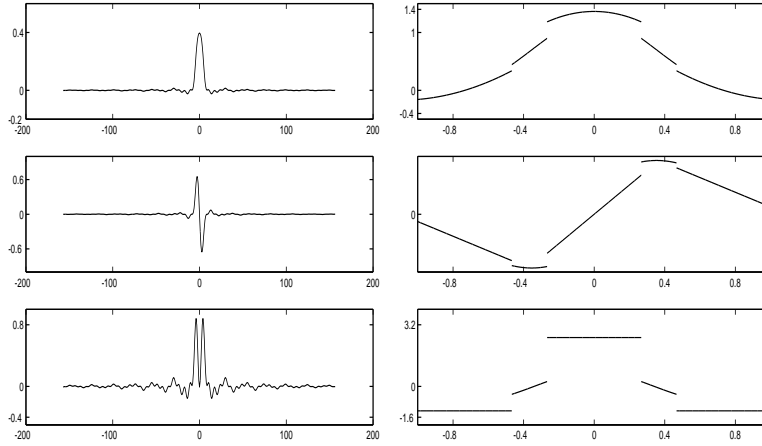


Figure 6:  $\Phi^*$  for  $h = \frac{11}{15}\phi$

Figure 7:  $\widehat{\varphi}_1^*, \frac{1}{i}\widehat{\varphi}_2^*, \widehat{\varphi}_3^*$ ;  $h = \frac{11}{15}\phi$

Figure 8 and Figure 9 show the error in the cases  $h = \frac{2}{3}\phi$  and  $h = \frac{11}{15}\phi$ , respectively. Notice that in the first case the order of magnitude of the error is  $10^{-4}$  while in the second case is  $10^{-3}$ . In second case, since the functions  $\varphi_j^*$  were computed numerically, to compute their values at the points  $x - kt_o$  we used spline interpolation. This accounts for the different order of magnitude of the error.

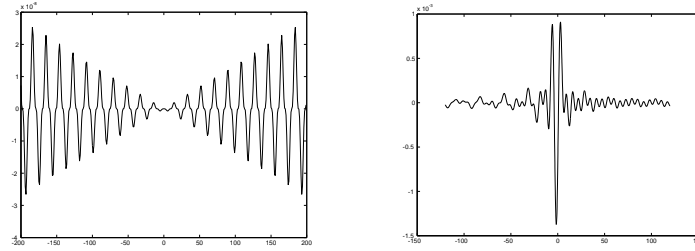


Figure 8: The error with  $h = \frac{2}{3}$

Figure 9: The error with  $h = \frac{11}{15}$

## ACKNOWLEDGMENTS

The author thanks Aldo Conca for helpful discussions on the Moore-Penrose inverse of a matrix.

## REFERENCES

- [AG] A. Aldroubi and K. Gröchenig *Non-Uniform Sampling and Reconstruction in Shift-Invariant Spaces*, *SIAM Rev.*, **43**, N.4, 585-620, 2001.
- [BH] M.G. Beaty and J.R. Higgins *Aliasing and Poisson Summation in the Sampling Theory of Paley-Wiener Spaces*, *J. Fourier Anal. Appl.*, **1**, N.1, 67-85, 1994.
- [BIG] A. Ben-Israel and T.N.E. Greville *Generalized Inverse. Theory and Applications*, John Wiley & Sons, 1974.
- [B] M. Bownik *The structure of shift invariant subspaces of  $L^2(\mathbb{R}^n)$* , *J. Funct. Anal.*, **177**, 2000, 282-309.
- [BDR] C. de Boor, R DeVore and A. Ron *The Structure of Finitely Generated Shift Invariant Spaces in  $L_2(\mathbb{R}^d)$* , *J. Funct. Anal.*, **119**, 1994, 37-78.
- [F] P.J.S.G. Ferreira *Mathematics for Multimedia Signal Processing II. Discrete Finite Frames and Signal Reconstruction*, Signal Processing for Multimedia, J. S. Byrnes (Ed.). IOS Press, pp. 35-54, 1999.
- [JF] D. Jagerman and L. Fogel *Some general aspects of the sampling theorem*, *IEEE Trans. Inform. Theory*, **2**, 1956, 139-156.
- [H] H. Helson *Lectures on Invariant Subspaces*, Academic Press, New York, 1964.
- [HW] C.E. Heil and D.F. Walnut *Continuous and discrete wavelet transforms*, *SIAM Rev.*, **31**, N.4, 628-666.
- [Hi] J.R. Higgins *Sampling Theory in Fourier and Signal Analysis. Foundations*, Oxford University Press, Oxford, 1996.
- [Hi1] J.R. Higgins *Sampling Theory for Paley Wiener spaces in the Riesz basis setting*, *Proc. Roy. Irish Acad. Sect. A* **94**, N.2, 219-236.
- [HS] J.R. Higgins and R. L. Stens *Sampling Theory in Fourier and Signal Analysis. Advanced Topics*, Oxford University Press, Oxford, 1999.
- [L] D.A. Linden *A discussion of sampling theorems*, *Proc. IRE*, **47**, 1959, 1219-1226.
- [LA] D.A. Linden and N.M. Abramson *A generalization of the sampling theorems*, *Inform. Control.*, **3**, 1960, 26-31.
- [P] A. Papoulis *Generalized sampling expansions. Circuit and Systems*, **24**, N.11, 1977, 652-654.
- [RS] A. Ron and Z. Shen *Frames and Stable Bases for Shift-Invariant Subspaces of  $L^2(\mathbb{R})$* , *Canad. J. Math.*, **47**, 1995, 1051-1094.
- [SF] D.M.S. Santos and P.J.S.G. Ferreira *Reconstruction from missing function and derivative samples and oversampled filter banks* in Proceedings of the IEEE International Conference on Acoustics, Speech, and Signal Processing, ICASSP 04, vol. 3, 2004, 941-944.
- [S] T.P. Srinivasan *Doubly-invariant subspaces*, *Pacific J. Math.* **14**, N. 2, 1964, 701-707.

DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI GENOVA, VIA DODECANESO 35, 16146 GENOVA, ITALIA